UNIVERSIDADE FEDERAL DO ESPÍRITO SANTO
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PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA

TAYS MIRANDA DE ANDRADE

The very early universe: An analysis on inflationary and collapsing scenarios.

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Tese apresentada ao colegiado do Programa de Pós- Graduação em Física da Universidade Federal do Espírito Santo como requisito parcial para a obtenção do título de Doutor em Física.

Orientador: Prof. Dr. Oliver Fabio Piattella

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"The very early universe: An analysis on inflationary and collapsing scenarios"

Tays Miranda de Andrade

Tese submetida ao Programa de Pós-Graduação em Física da Universidade Federal do Espírito Santo, por webconferência, utilizando MConf, como requisito parcial para a obtenção do título de Doutor em Física.

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Resumo

A inflação é uma expansão acelerada que pode ter ocorrido a uma energia muito alta no Universo primordial. Além de trazer soluções para os problemas do modelo de big bang quente, ela também fornece um mecanismo efetivo para a produção das flutuações cosmológicas, as quais geram as estruturas em larga escala como vemos atualmente. Entretanto é importante evidenciar que a inflação não é o único mecanismo capaz de resolver esses problemas de condições iniciais do nosso universo. Cenários colapsantes podem ser uma opção alternativa para originar o big bang quente e as perturbações primordiais. Na primeira parte desta tese, nós apresentamos o cenário inflacionário, com ênfase no modelo de Starobinsky, discutindo sua configuração geral, predições e uma possível modificação à inflação de R^2 motivados pelos atratores-α. A segunda parte é dedicada à análise de cenários colapsantes, baseados em um campo escalar e um potencial exponencial, com a inclusão do formalismo estocástico, o qual nos permite calcular o barulho estocástico gerado por flutuações quânticas cruzando uma escala de corte para o campo escalar no limite super-Hubble.

Palavras-chave

Inflação, modelo de Starobinsky, campo escalar, potenciais exponenciais, universo colapsante.

Abstract

Inflation is an accelerated expansion that may occurred at very high energy in the early Universe. In addition to bringing a solution for the hot big bang model problems, it also provides an effective mechanism for the production of cosmological fluctuations, which give rise to the large scale structures as we see today. However, it is important to highlight that inflation is not the only mechanism capable of solving the initial issues of our Universe. Collapsing scenarios can be an alternative option to originate the hot Big Bang and the primordial perturbations. In the first part of this thesis, we present the inflationary scenario, with emphasis on the Starobinsky model, discussing its general set-ups, predictions and a possible modification to the R^2 inflation motivated by the α-Attractors. The second part is dedicated to the analysis of collapsing scenarios, based on a scalar field with exponential potential with inclusion of the stochastic formalism, which allows us to compute the stochastic noise generated by the quantum fluctuations crossing into a coarse grained, super-Hubble scalar field.

Keywords

Inflation, Starobinsky model, scalar field, exponential potentials, collapsing universe.

Acknowledgements

I would like to start this section with three sentences that defined my PhD, that I will certainly pass on to my future students, and can also serve as a stimulus for eventual PhD students who come to read this thesis:

"One *epsilon* at a time and at the end of the day, we will have a reasonable amount" (Oliver Piattella, 2016).

"Make it simple" (David Wands, 2018).

"A written page is always worth more than a blank page" (Júlio Fabris, 2015).

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"Se as coisas são inatingíveis... ora!
Não é motivo para não querê-las...
Que tristes os caminhos, se não fora
A presença distante das estrelas!"

Mário Quintana

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1 Introduction

Cosmic Inflation [6, 7, 8, 9, 10, 11, 12], which is a phase of accelerated expansion that occurred in the early Universe, has an important place in Modern Cosmology. It was first invoked due to its ability to solve some of the puzzles plaguing the hot Big Bang model (e.g. the flatness and the horizon problems). Moreover, it did not take much time to realise that inflation could be a possible explanation for the seeds which generate the large scale structures in our Universe. Indeed, an initial power spectrum due to quantum vacuum fluctuations can be originated during inflation by using a simple model: a self interacting and slowly rolling scalar field. So, given its ability to make predictions on the primordial power spectrum, which can be tested through observation of the cosmic background radiation (CMB) [13, 1], inflation has become a very active field of research nowadays.

Since the first inflationary solution, proposed by Starobinsky in [6], many other proposals have been put forward. These have been classified and statistically analysed in many works, see e.g. [14, 15, 16, 17, 18, 19], and the result is that data favour the simplest category of inflationary models: single-field slow-roll inflation, with a plateau potential. The typical representative of this class of models is the Starobinsky model, which can be expressed within the framework of $f(R)$ theories [20] as the following quadratic correction to the Einstein-Hilbert action

$$f(R) = R + \frac{R^2}{6M^2}\,, \tag{1.1}$$

where M is an arbitrary energy scale, typically $M \sim 10^{13}$ GeV [21].

The predictions of the Starobinsky model on the fundamental inflationary parameters, i.e. the scalar spectral index n_s and the tensor-to-scalar ratio r are in very good agreement with the latest Planck results [1]. However, there is a looseness in r due to our current inability to detect the B-modes of CMB polarisation, which carry crucial information about the primordial gravitational wave background. This looseness might be threatening to the primacy of the Starobinsky model if, for example, a sufficiently large r was eventually detected. In this case the Starobinsky model would be ruled out. In this sense it is interesting to explore new theoretical scenarios which could allow more freedom to tune the gravitational wave production maintaining the behavior of $n_{\rm s}$, as it happens for the Starobinsky model.

As an example of these scenarios, recently a new class of inflationary models dubbed α-Attractors has been proposed in [22, 23, 24]. Remarkably, the predictions for this class of models are very similar to those of the Starobinsky model, so that the excellent agreement with observations for the parameter $n_{\rm s}$ is maintained, while there is freedom to tune the gravitational wave production through the parameter α.

It is the first purpose of this thesis to answer the following question: how should we modify the Starobinsky model in the $f(R)$ framework in order to have a larger production of gravitational waves? Therefore, we reconstruct a $f(R)$ theory which could mimic the α-Attractors scalar potential and do an extensive study of it, as shown in Chapter 4.

On the other hand, we also are interested on investigating another possibility that could explain the origin of primordial perturbations of our Universe: the case in which they originate before the Big Bang. Since there is no single agreed model for which a fundamental field is responsible for driving inflation and generating structure, we also can construct theories in which the large scale structure of our Universe could be inherited from vacuum fluctuations during an earlier phase, before the Big Bang. In this context, another possibility to generate the primordial perturbations would be a preceding collapse. The first proposal in which the collapse phase could set the initial conditions for a subsequent post-big bang phase was made by Gasperini and Veneziano [25]. Since then, several models have been presented, see e.g. [26, 5, 27].

It has been investigated before that a scalar field and an exponential potential can drive an accelerated expansion and thus provide possible models for inflation in the early universe [28, 29, 30] or dark energy at the present epoch [31, 32]. It has been also shown that the same mechanism can drive a collapsing universe too, and this configuration is particularly interesting since due to their scale-invariant form, exponential potentials are simple to study analytically. In order to explore this simplicity, we will focus our attention on models with a scalar field φ

and scalar potential

$$V = V_0 \exp\left(-\kappa\lambda\varphi\right) , \tag{1.2}$$

where $\kappa = \sqrt{8\pi G}$ and λ is the slope of the potential. Although the classical stability of collapsing models has already been studied in previous works [5], it becomes of paramount importance to consider how the classical solutions behave in the presence of quantum fluctuations. To do this we will extend the stochastic formalism, previously introduced to study inflation, to collapse scenarios. This approach models the effect of quantum vacuum fluctuations by introducing a cut-off scale (the so-called coarse-graining scale) splitting the fluctuations into two parts: quantum vacuum modes (below the coarse-graining scale) and the long wavelength field which includes a stochastic noise. It was introduced in cosmology by Starobinsky [33] to describe the effects of random vacuum fluctuations on inflationary dynamics, see e.g. [34, 35, 36, 37, 38, 39]. More recently, this formalism was used in [40] to show that slow-roll inflation is a stochastic attractor.

The presence of a stochastic noise can affect the stability of a dynamical system in a non-trivial way. The second purpose of this thesis is to investigate how these stochastic perturbations can modify the equation of state of the inflationary or collapse cosmology, as shown in Chapter 5.

Although our work aims to analyse physics of the very early universe, it is more convenient to start our study by describing the universe today, since the late universe can provide a wide range of information about its beginning. So, we shall begin this thesis by presenting the Standard Model of Cosmology, and discussing problems that arise with the initial conditions of evolution of the universe - the horizon and flatness problems in Chapter 2. Having presented the problems, we will begin Chapter 3 which proposes inflation as solution of these problems. We will describe the dynamics of the scalar field that would drive the inflationary phase and present its predictions. Sections 4 and 5 are dedicated to the papers published during this thesis, [41, 42, 43]. Finally, in a last section, we present some concluding remarks and possible prospects for the present work.

Throughout this thesis we will use the convention of natural units: $\hbar = 1$, $k_B = 1$ and $c = 1$.

2

ΛCDM: An Introduction to The Cosmological Standard Model

In the precision era, the high accuracy observational data such as CMB measurements, galaxy and supernova surveys, 21 cm astrophysics data, gravitational waves detectors and others have put Cosmology as a genuine science, since we are able to make and test theoretical predictions. We start this chapter revisiting the basis of the Cosmological Standard Model, which is discussed in references such as [44, 45, 46, 47, 48]. We will give a brief overview of this model, and arise questions left by it, which can be solved by introducing an accelerated expansion right after the Big Bang.

We have accurate observational evidence that the current Cosmological Standard Model, known as ΛCDM (Lambda Cold Dark Matter), describes the evolution of our Universe since an initial singularity 13.8 billion years ago. This model is based on the Cosmological Principle and has two exotic components: dark matter and dark energy (in the form of a cosmological constant). It is based on three observational pillars [49]

- The Hubble diagram, showing the expansion of the universe [50];
- Light elements abundance, according to the nucleosynthesis theory suggested initially by Gamow [51];
- The detection of cosmic microwave background radiation (CMB) as a spectrum of black body, which gives us information about a very early time in the universe and proves its homogeneity [52];

Geometry

The cosmological principle symmetries, which state that the Universe is isotropic and homogeneous at large scales [$\sim O(100)$ Mpc], fix the form of the metric that encodes the geometry of the spacetime as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\mathcal{K}r^2} + r^2 d\Omega^2\right] , \tag{2.1}$$

where $a(t)$ is the scalar factor, $\mathcal{K} = -1, 0, 1$ is the curvature parameter and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard metric on the 2-sphere. In this metric, t is the cosmic time, r is the comoving radial coordinate, θ and ϕ are the comoving angular coordinates.

As physical results depend on the physical coordinate, $r_{phys} = a(t)r$, we calculate the physical velocity of an object as

$$v_{phys} = \frac{dr_{phys}}{dt} = a\frac{dr}{dt} + r\frac{da}{dt} \tag{2.2}$$

$$\equiv v_{pec} + Hr_{phys} , \tag{2.3}$$

where v_{pec} is the peculiar velocity of the object, which is the velocity measured by an observer who follows the Hubble flow, and $H = \dot{a}/a$ is the Hubble parameter. If we neglect the peculiar velocity, this becomes the Hubble-Lemaître law and its first observation was in 1929 [50]. The current value of H is denoted by H_0, and it contains important information on our Universe since it is related to the age of the Universe and to the scale of the size of the observable Universe. An issue for cosmologists now is that current observations do not agree on the value of the parameter H_0: In 2019, Reid et al. [53] showed the local measurement of the Hubble constant is $H_0 = 75 \pm 1.4 \mathrm{km\ s^{-1}\ Mpc^{-1}}$; which is higher than $H_0 = 67.4 \pm 0.5 \mathrm{km\ s^{-1}\ Mpc^{-1}}$ given by the Planck Collaboration [54].

In a FRLW spacetime, light travels along geodesics with $ds = 0$ towards the observer at rest[1]. As we observe today electromagnetic waves emitted in different periods of the universe,

[1]The universe is homogeneous and isotropic, so the observer should not measure any spacial velocity, $d\theta = d\phi = 0$.

light gets redshifted because of the time dependence of the scale factor. So it is convenient to define the redshift parameter

$$1 + z = \frac{\nu_1}{\nu_0} = \frac{a(t_0)}{a(t_1)} \,, \tag{2.4}$$

where z is the redshift parameter, and we are considering that light was emitted from its source at time t_1 with frequency ν_1, and received by the observer at t_0 with frequency ν_0. If we think in terms of wavelengths, it is clear that the wavelength of light contracts and stretches according to the scale factor $\lambda \propto a$.

By introducing the conformal time

$$d\eta = \frac{dt}{a(t)} \,, \tag{2.5}$$

equation (2.1) can be written in the comoving frame as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1 - \mathcal{K}r^2} + r^2 d\Omega^2\right] \,, \tag{2.6}$$

which means that, in this form, we have a static metric multiplied by the conformal factor $a^2(\tau)$.

Dynamics

The dynamics of the gravitational field, which relates the geometry of spacetime with its material content, is described by the action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + \int d^4x\sqrt{-g}\mathcal{L}_m \,, \tag{2.7}$$

where $\kappa = 8\pi G = 1/M_{Pl}^2$ is the coupling constant, g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, Λ is the cosmological constant and $\mathcal{L}_m$ is the lagrangian density which describes the matter content.

By varying the above action with respect to $g_{\mu\nu}$, we obtain the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \ , \tag{2.8}$$

Since we are considering scales above 100 Mpc, the matter content of the universe, represented by $T_{\mu\nu}$, can be seen as a perfect fluid

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \ , \tag{2.9}$$

where ρ is the energy density, P is pressure and u_μ is the four-velocity of the fluid.

When we apply the geometry given by (2.1) into the components of (2.8), we obtain the two following dynamical equations[2]

$$H^2 = \frac{\kappa\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \ , \tag{2.10}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3P) + \frac{\Lambda}{3} \ , \tag{2.11}$$

which are known as the Friedmann and the Raychaudhuri equations, respectively. The former shows a relation between the change of the scale factor and energy density, spatial curvature and cosmological constant of the universe. Since in this thesis we are focused in the study of the very early universe, k and Λ can be neglected, and the new Friedmann equation states that the expansion (or contraction) of the universe is governed by the energy density, only. From the latter, we can see that in absence of a cosmological constant, there is an accelerated expansion only if we violate the strong energy condition, *i.e.*, $\rho + 3P < 0$. This is the root of the inflationary scenario and will be studied in more details in Chapter 3.

[2]From the four equations found, only two are independents. The three equations with spatial components are equivalent, as they reflect the isotropy of the metric.

Composition

Equations (2.10) and (2.11) when combined give rise to the conservation equation

$$\dot{\rho} + 3H(\rho + P) = 0 \,, \tag{2.12}$$

which may also be derived from the conservation of energy-momentum $\nabla_\nu T^{\mu\nu} = 0$.

We will solve the conservation equation by using the *ansatz* $P = w\rho$, with w constant. Its solution is

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \quad \rightarrow \quad a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \quad . \tag{2.13}$$

By using w constant, we were able to solve analytically (2.12) keeping a realistic approach, since there are three particular values of w which play an important role in the evolution of our Universe. They are

- Radiation: $w_r = 1/3 \rightarrow \rho_r \propto a^{-4}$ and $a \propto t^{1/2}$;
- Cold matter: $w_b = 0 \rightarrow \rho_b \propto a^{-3}$ and $a \propto t^{2/3}$;
- Dark energy: $w_\Lambda = -1 \rightarrow \rho_\Lambda \propto a^0$ and $a \propto e^{Ht}$;

In an universe in which the total energy density contains: radiation (photons and neutrinos) ρ_r; baryonic matter (also known as cold matter or dust, since it is non relativistic) ρ_b; dark matter ρ_c; and dark energy ρ_Λ; Equation (2.10) becomes[3]

$$H^2 = H_0^2 \left(\frac{\Omega_r}{a^4} + \frac{\Omega_b}{a^3} + \frac{\Omega_c}{a^3} + \frac{\Omega_\nu^{\mathrm{nrel}}}{a^3} + \Omega_\Lambda + \frac{\Omega_k}{a^2}\right) \,, \tag{2.14}$$

[3]Note also that the spatial curvature can be identified as a fluid with $w = 1/3$.

where the dimensionless density parameter Ω for each one of the four first components, and the density parameter Ω for curvature are defined by, respectively

$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i \quad \text{with} \quad i = r, b, c, \Lambda \ , \quad \text{and} \quad \Omega_k = -\frac{k}{a^2H^2} \ . \tag{2.15}$$

The definition of critical density for a flat universe ($k = 0$) is

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G} \ , \tag{2.16}$$

and we could relate the critical density to the density parameter by

$$\Omega_i = \frac{\rho_i}{\rho_{\text{crit}}} \ . \tag{2.17}$$

Equation (2.14) is the Friedmann equation for the most successful model in Cosmology, the ΛCDM, which is made of dark energy Λ, cold dark matter[4], ordinary matter and radiation. From Planck 2018 [54], the up-to-date values for the current density parameters of each component are: $\Omega_{\Lambda,0} \sim 70\%$; $\Omega_{c,0} \sim 25\%$; $\Omega_{b,0} \sim 5\%$; $\Omega_{r,0} + \Omega_{\nu,0} \sim 0.01\%$;

Current data still indicates that the universe is spatially flat, since we have a small density parameter

$$\Omega_{k,o} = 0.0007^{+0.0037}_{-0.0037} \ , \tag{2.18}$$

at the 95% confidence level.

We treated radiation, baryonic matter and dark energy as a perfect conservative fluid, so the temporal evolution of each of them (with respect to the scale factor) is obtained by the solution of (2.13), and this allows us to see in which phase of the universe each one of them dominated. By considering the standard cosmological model, after inflation, the universe had

[4]Although it has a distinct nature from ordinary matter, dark matter also is treated as cold, which means it has no pressure, so its parameter of state is given by $w = 0$. This makes impossible to distinguish both of them at background level.

a phase of domination of radiation, followed by a phase of domination of matter until reaching a current phase of predominance of dark energy.

Also, it is important to highlight that the Universe has inhomogeneities. From CMB observations [55, 56], we know that there are temperature fluctuations $\delta T/T \sim O(10^{-5})$. These tiny fluctuations confirm the Universe is homogeneous and isotropic to at least one part in $100,000$. However, these deviations are a fundamental prediction of the inflationary cosmology, as we will see in Chapter 3.

The problems of the hot Big Bang model

In principle, we would like to describe a physical system given its initial conditions, but it is somewhat philosophical to question whether the initial conditions necessary for evolution of the system are contained in the theory or they are separate from it. For example, in Classical Mechanics, we know that given the initial position and velocity, we can use the laws of Newton to describe the trajectory of a body. But we still do not know if Cosmology has as a role in predicting or even explaining the primary conditions required. Obviously, we want to avoid imposing very restrictive initial conditions to observe the universe as it is today. So in this section, based on [57], we will describe two problems the usual Big Bang theory faces for the universe to evolve to its current state, and we will show that these problems can be solved if we consider the universe had a phase of quasi de Sitter[5], right after leaving its quantum stage, which is the main idea of inflation.

[5] It is important to note that this is only a phase because if the Hubble parameter does not evolve over time, inflation continues indefinitely, making the universe a huge emptiness with very low temperatures - a pure de Sitter universe, but of course this is in contradiction with current observations. Then, inflation has to end and the vacuum energy must turn into thermal energy for the universe. This mechanism is known as reheating [58, 59], and it consists of the decay of vacuum energy in particles, resulting in the increase of entropy of the universe and the beginning of the era dominated by radiation.

The horizon problem

This is a causality problem and could be reformulated as the following question: Why CMB is so uniform?

The photons we see today are those emitted from the surface of last scattering (380.000 years ago). The angular diameter distance d_A on the last scattering surface is given by

$$d_A(t) = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')} \tag{2.19}$$

$$d_A(t_0) \sim H_0^{-1}(1+z_L)^{-1} \; . \tag{2.20}$$

Knowing that the horizon size in an universe dominated by matter and radiation at decoupling time is

$$d_H(t_{dec}) \sim H_0^{-1}(1+z_L)^{-3/2} \; , \tag{2.21}$$

since $a(t)$ grows as $t^{2/3}$ in the epoch of the last scattering, the ratio between the particle horizon at the decoupling time and the particle horizon today is $d_H(t_{dec})/d_A(t_0) \sim 1.6^\circ$, for $z_L \sim 1100$. This means that in an universe dominated by matter or radiation, no physical influence could have smoothed out initial inhomogeneities and brought about the almost same temperature points that were separated by more than 2° initially.

If we suppose the universe had an inflationary phase before the radiation era, with $H \approx$ constant, the scalar factor behaves as

$$a(t) = a_i e^{H_I(t-t_i)} \;\rightarrow\; a(t) = a_I e^{-H_I(t_I - t)} \; , \tag{2.22}$$

where the subscripts "i" and "I" represent the beginning and end of the inflationary era, respectively. Now, (2.19) becomes

$$d_H(t_L) = a(t) \int_{t_i}^{t_L} dt' \frac{1}{a_I} e^{H_I(t_I - t')} = \frac{a(t_L)}{a_I H_I}[e^N - 1] \; , \tag{2.23}$$

were $N = H_I(t_L - t_i)$ is the number of e-folds of expansion during inflation. In order to solve the horizon problem, $e^N \gg 1$, so we can drop the term -1 in square brackets.

To satisfy the high degree of isotropy presented by CMB at large scales, we need

$$d_H(t_L) > d_A(t_L) \,, \tag{2.24}$$

being $d_A \approx a/H_0$ for small scale factors.

The condition (2.24) for the isotropy of CMB is then

$$e^N > \frac{a_I H_I}{a_0 H_0} \,. \tag{2.25}$$

The flatness problem

The non relativistic and relativistic contributions to the total energy density of the universe ρ grow as a^{-3} and a^{-4}, respectively, so in the beginning of the expansion, we can neglect k, and the Friedmann equation becomes

$$H \to \frac{\kappa}{3}\rho \,, \tag{2.26}$$

i.e., in the primordial universe the density of the material content is equivalent to the critical density $3H^2/\kappa$.

On the other hand, data from the redshift-distance relation of observed supernovae Type Ia, age measurements of old stars and CMB temperature fluctuations data favor a curvature density $\Omega_k \approx 0$. So, using the constraint from the Friedmann equation

$$\Omega_T + \Omega_k = 1 \,, \tag{2.27}$$

where $\Omega_T = \Omega_r + \Omega_c + \Omega_b + \Omega_\Lambda$, we would expect $|\Omega_k| < 1$, but $\Omega_k = -k/a^2H^2 = k/\dot{a}^2$, which increases with time.

At the time when there is domination of non-relativistic matter (from the stage when the universe temperature dropped from 10^4K until close to the present time), $a \propto t^{2/3}$, so $|k|/\dot{a}^2$ also grows with $t^{2/3} \propto T^{-1}$. So if we consider $|\Omega_k| < 1$, when the universe had a temperature of 10^4K, the curvature parameter had to be of the order of 10^{-4}. If we go back more in the history of the universe and consider the era of radiation, $|k|/\dot{a}^2$ grows with $t \propto T^{-2}$. Again, for $|k|/\dot{a}^2$ not to be less than 10^{-4} when $T = 10^4$K, it would be necessary that $|k|/\dot{a}^2$ was of the order of 10^{-16} in the primordial nucleosynthesis phase and even lower in older times. This means that a flat universe today was much more flat in the past. This would require a huge fine tuning of the spatial curvature of the primordial universe.

In order to solve this problem, we will consider the inflationary phase, as we did in the horizon problem.

If $|k|/a_i^2 H_i^2 \sim O(1)$ in the beginning of inflation, in the end it would be of order

$$\frac{|k|}{a_I^2 H_I^2} = e^{-2N} \frac{|k|}{a_i^2 H_i^2} \approx e^{-2N} \ . \tag{2.28}$$

Today ($a_0 = 1$) we would have

$$|\Omega_{k,0}| = \frac{|k|}{H_0^2} = \frac{|k|}{a_I^2 H_I^2} \left(\frac{a_I^2 H_I^2}{H_0^2} \right) = e^{-2N} \frac{a_I^2 H_I^2}{H_0^2} < 1 \ . \tag{2.29}$$

So the flatness problem would be avoided if the expansion during inflation had a limit given by

$$e^N > \frac{a_I H_I}{H_0} \ , \tag{2.30}$$

which matches the condition to solve the horizon problem.

3

Review of Cosmological Inflation

In this chapter, we will review some of the fundamental aspects of inflation by showing a simple, but powerful configuration which generates inflation, and its predictions, based in references such as [44, 45, 46, 47, 48, 49, 60, 57]. Our main goal is to discuss the context which motivate our results presented in Sec.4.

Single field inflation

It is clear to see that the condition $\ddot{a} > 0$ solves the Hot Big Bang problems presented before. Now the question is: what kind of matter component could drive such an accelerated phase?

The inflaton

A simple and successful way to achieve inflation is by considering a quantum scalar field (named inflaton) rolling slowly down its potential. The action of such a system in a general cosmological background is given by

$$S = \int \sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] . \tag{3.1}$$

The energy-momentum tensor of φ is

$$T^{\varphi}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + V(\varphi)\right] . \tag{3.2}$$

By considering the spatial curvature $k = 0$ in a FLRW spacetime, the energy density ρ_φ and the pressure P_φ of the scalar field are given by[1]

$$\rho_\varphi = -T_0^{0(\varphi)} = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \; , \tag{3.3}$$

$$P_\varphi = T_i^{i(\varphi)} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \; . \tag{3.4}$$

Thus, the parameter w of the equation of state for φ is given by

$$w = \frac{P_\varphi}{\rho_\varphi} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)} \; . \tag{3.5}$$

We already know that inflation requires an accelerated expansion of the universe, and this condition can be fulfilled in the case of a scalar field if

$$V(\varphi) \gg \dot{\varphi}^2. \tag{3.6}$$

To obtain $w \backsimeq -1$, the function $V(\phi)$ can be left unspecified. In fact, the origin of the inflaton is still unknown, and there are several possibilities for the shape of $V(\varphi)$[2]. Going further, condition (3.6) shows that when the potential energy dominates over its kinetic energy, the inflaton slowly rolls down its potential and inflation is achieved. For this to happen, the inflaton potential also must be sufficiently flat, which could be a challenge in the context of string theory or supergravity [61, 62].

The inflationary dynamics can be obtained by varying action (3.1) in terms of φ. Its equation of motion is

$$\frac{d^2\varphi}{dt^2} + 3H\frac{d\varphi}{dt} + \frac{\partial V(\varphi)}{\partial \varphi} = 0 \; , \tag{3.7}$$

[1] We are considering a homogeneous field and setting $\nabla\varphi/a \to 0$, since inflation rapidly smooths out the spatial variation.

[2] One of the most important issues in inflationary cosmology is how to distinguish different models which provide almost the same predictions on the inflationary parameters.

which is the Klein-Gordon equation, and it is subject to the Friedmann constraint

$$3H^2 = \frac{\rho_\varphi}{M_{Pl}^2} \,. \tag{3.8}$$

Here, it is important to show that by inserting the Klein-Gordon equation into the time derivative of the Friedmann constraint, we obtain a new way to write the acceleration equation

$$\dot{H} = -\frac{1}{2M_{Pl}^2}\dot{\varphi}^2 \,, \tag{3.9}$$

which can be very useful in this context.

From (3.7), it is clear that the potential acts like a force ($\partial V/\partial\varphi$), and in the second term, the expansion of the universe adds friction, increasing the time that the scalar field takes to reach the minimum of its potential.

The condition (3.6) assures that H is nearly constant

$$V(\varphi) \gg \dot{\varphi}^2 \to H^2 \backsimeq \frac{V(\varphi)}{3M_{Pl}^2} \,, \tag{3.10}$$

leading to quasi-exponential expansion $a \sim e^{Ht}$. Also, to assure inflation is prolonged enough time, we need the second condition

$$\ddot{\varphi} \ll 3H\dot{\varphi} \;\to 3H\dot{\varphi} \backsimeq -\frac{\partial V}{\partial \varphi} \,, \tag{3.11}$$

which allowed us to drop the double derivative in (3.7).

These conditions are equivalent to requiring the slow roll parameters [63, 60]

$$\epsilon_H = -\frac{\dot{H}}{H^2} \,, \quad \eta_H = -\frac{\ddot{\varphi}}{H\dot{\varphi}} \,, \quad \xi_H = \frac{\dot{\epsilon}_H - \dot{\eta}_H}{H} \,, \tag{3.12}$$

to be small $\epsilon,\ \eta,\ \xi \ll 1$. The slow roll parameters are fundamental quantities, since they enable us to compute straightforwardly the scalar index, n_s, the tensor-to-scalar ratio, r, and the running of the spectral index, α_s.

The amount of inflation is quantified by the number of e-folds

$$\Delta N = \int_{t_i}^{t_f} H dt = -\frac{1}{M_{Pl}^2} \int_{\varphi_i}^{\varphi_f} \frac{\partial \phi}{\partial V} V(\phi) d\phi \ , \tag{3.13}$$

where $\Delta N = N_f - N_i$, and φ_f is the value of the field at the end of inflation, which is found by the condition $\epsilon(\varphi_f) \approx 1$ (or, $\eta(\varphi_f) \approx 1$).

We know that the total number of inflationary e-folds should exceed about 60 in order to solve the horizon and flatness problems. The precise value depends on the energy scale of inflation and on the details of reheating after inflation. The fluctuations observed in the CMB are created during approximately $N_{\rm CMB} \backsimeq 50 - 60$ e-foldings before the end of inflation. Therefore, we use these numbers to have a precise prediction on the values of the slow-roll parameters and then on the spectral index.

An interesting way to know if a given potential $V(\varphi)$ is suitable to generate inflation (in the slow roll regime) is to compute the *potential* slow roll parameters, which are related to the Hubble parameters by

$$\epsilon_V \approx \epsilon_H \ , \quad \text{where} \quad \epsilon_{\rm V} \equiv \frac{\rm M_{Pl}^2}{2} \left(\frac{\rm V_\varphi}{\rm V} \right)^2 \ , \tag{3.14}$$

$$\eta_V \approx \epsilon_H + \eta_H \ , \quad \text{where} \quad \eta_{\rm V} = \rm M_{Pl}^2 \frac{V_{\varphi\varphi}}{V} \ , \tag{3.15}$$

$$\xi_V \approx \xi_H + 2\epsilon_H (3\eta_H - \epsilon_H) \ , \quad \text{where} \quad \xi_{\rm V} \equiv \rm M_{Pl}^4 \frac{V_\varphi V_{\varphi\varphi\varphi}}{V^2} \ , \tag{3.16}$$

here, $V_\varphi = \partial V / \partial \varphi$.

Cosmological perturbations

In addition to solving the horizon and flatness problems, cosmic inflation is also recognised for being the mechanism from which the primordial perturbations present on large scales in our Universe are generated. This happens because during inflation, both the scalar and metric fields undergo quantum-mechanical fluctuations, and observational consequences of the inflationary scenario can be derived [64, 65, 66, 67, 68, 69]. In this section, we obtain the observables $\mathcal{P}_\zeta$, n_s, r, and α_s from the dynamics of the inflationary field. Since scalar and tensor perturbations are generated in a similar way, we shall begin with a qualitative description of the effects of the former and then, extend it to the latter.

Equation of Motion for the Perturbations

We will derive the inflationary spectrum of a simple model with a scalar field in an intuitive way. To start, let us break the homogeneity assumption of φ by considering

$$\varphi = \varphi(t) + \delta\varphi(x_i, t) \, , \tag{3.17}$$

where $\delta\varphi(x_i, t)$ are small inhomogeneities.

Taking into consideration that a small perturbation $\delta\varphi(x_i, t)$ induces metric perturbations

$$ds^2 = -(1+2A)dt^2 + 2a(t)B_{,i}dx^i dt + a^2(t)\left[(1-2\psi)\delta_{ij} + 2E_{,ij} + h_{ij}\right] dx^i dx^j \, , \tag{3.18}$$

we find that, at linear order in the scalar metric perturbations and $\delta\varphi$, the Klein-Gordon equation (3.7) becomes

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + \frac{k^2}{a^2}\delta\varphi + V_{\varphi\varphi}\delta\varphi = -2AV_\varphi + \dot{\varphi}\left[\dot{A} + 3\dot{\psi} + \frac{k^2}{a^2}\left(a^2\dot{E} - aB\right)\right] \, , \tag{3.19}$$

where k is the comoving wavenumber.

In order to simplify (3.19), we can use the relations between the Einstein equations and the scalar metric perturbations, which are obtained by the 00 and $0i$ components

$$3H(\dot{\psi} + HA) + \frac{k^2}{a^2}\left[\psi + H(a^2E - aB)\right] = -4\pi G\delta\rho \tag{3.20}$$

$$\dot{\psi} + HA = -4\pi G\delta q \; . \tag{3.21}$$

These equations are also known as the energy and momentum constraints [66, 27]. The energy and pressure perturbations and momentum are computed as

$$\delta\rho = \dot{\varphi}(\dot{\delta\varphi} + A\dot{\varphi}) + V_\varphi\delta\varphi \tag{3.22}$$

$$\delta P = \dot{\varphi}(\dot{\delta\varphi} + A\dot{\varphi}) - V_\varphi\delta\varphi \tag{3.23}$$

$$\delta q = -\dot{\varphi}\delta\varphi \; . \tag{3.24}$$

By using (3.20) and (3.21), it is possible to eliminate the metric perturbations from (3.19), and write a Klein-Gordon equation only in terms of the field perturbations as[3]

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \left[\frac{k^2}{a^2} + V_{\varphi\varphi} - \frac{8\pi G}{a^3}\left(\frac{a^3\dot{\varphi}^2}{H}\right)^{\dot{}}\right]\delta\varphi = 0. \tag{3.25}$$

It can be compactly written in terms of conformal time and $v = a\delta\varphi$ and $z = a\dot{\varphi}/H$ as

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0 \; , \tag{3.26}$$

which is known as the Mukhanov-Sasaki equation. This equation describes the dynamics of the whole scalar sector on both super ($k < aH$) and sub-horizon ($k > aH$) scales.

Note that when $\dot{\varphi}/H =$ constant (or equally ϵ_1 constant), (3.26) becomes

$$v_k'' + \omega^2(\eta, k)v_k = 0 \; , \tag{3.27}$$

[3] It is important to highlight that (3.25) is written in the spatially flat gauge ($\psi = 0$), which is a common gauge used to describe scalar perturbations during inflation. For more information on the variety of gauge-invariant combinations of the scalar metric perturbations, see [70, 71].

where $\omega^2 = k^2 - a''/a$. In eq.(3.27), each mode behaves as a parametric oscillator, and its solution for a given k can be expressed as

$$v_k = \sqrt{k|\eta|}\left[V_+ H_\nu^{(1)}(k|\eta|) + V_- H_\nu^{(2)}(k|\eta|)\right] , \tag{3.28}$$

where $H_\nu^{(1)}(k|\eta|)$, $H_\nu^{(2)}(k|\eta|)$ are Hankel functions of the first and second kind, and V_+, V_- are constants to be set by initial conditions.

In the remote past, all modes were inside the horizon, which means that all cosmological modes had time-independent frequencies, and under this approximation, the solution for the Mukhanov-Sasaki becomes

$$v_k = A_k e^{-ik\eta} + B_k e^{ik\eta} . \tag{3.29}$$

By using the Wronskian normalisation $W = v_k v_k'^{\star} - v_k' v_k^{\star} = i$, we obtain

$$W = 2ik\left(|A_k|^2 - |B_k|^2\right) = i . \tag{3.30}$$

The positive frequency mode $v_k \propto e^{-ik\eta}$ corresponds to the minimal excitation state. Then, we will choose this mode to define the inflationary vacuum state, which amounts to setting $A = 1/\sqrt{2k}$ and $B = 0$.

Then, we can set the initial condition as

$$\lim_{\eta \to -\infty} v_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} ; \tag{3.31}$$

which is known as the Bunch-Davies vacuum.

Now, we normalise the solution (3.28) with the Bunch-Davies vacuum on small scales (early times), so we set $V_+ = 0$ and $V_- = \sqrt{\pi/4k}$. Then, eq.(3.27) provides us with the corresponding solution on large scales (late times)

$$v_k = \frac{\sqrt{\pi|\eta|}}{2} H_\nu^{(2)}(k|\eta|) \; . \tag{3.32}$$

We can approximate the amplitude-squared (3.32) in the super-horizon limit as

$$\begin{aligned} v_k(\eta)v_k^\star(\eta) &\rightarrow \frac{1}{4\pi k^3}\frac{\Gamma^2(\nu)2^{2\nu}}{|\eta|^{2\nu-1}} \\ &\backsimeq \frac{1}{2k^3|\eta|^2} = \frac{(aH)^2}{2k^3} \; , \end{aligned} \tag{3.33}$$

where we considered the slow roll regime ($\nu \backsimeq 3/2$).

This enable us to define the power-spectrum of the field fluctuations as[4]

$$\frac{1}{a^2}\mathcal{P}_v(k,\eta) = \frac{H^2}{(2\pi)^2} \; , \tag{3.34}$$

which indicates a scale-invariant spectrum on super-horizon scales.

The power spectrum of the Mukhanov-Sasaki variable $\mathcal{P}_v$ can be directly related to the power spectrum of the comoving curvature perturbation $\mathcal{P}_\zeta$, and this change is convenient since ζ is a conserved quantity on large scales. It means that its spectrum can be propagated to the time of recombination without taking into consideration sensitive details of the cosmological evolution. The curvature perturbation is expressed in terms of the Mukhanov-Sasaki variable as

$$\zeta = \frac{v}{z} = \frac{v}{a\sqrt{2\epsilon_H}M_{Pl}} \; , \tag{3.35}$$

then, using (3.34), we obtain

$$\mathcal{P}_\zeta(k) = \frac{\mathcal{P}_v(k)}{2a^2\epsilon_H M_{Pl}^2} = \frac{V^3}{12\pi^2 M_{Pl}^6 V_\varphi^2} \; , \tag{3.36}$$

which is usually evaluated at some pivot scale $k_* = 0.05\mathrm{Mpc}^{-1}$.

[4]Taking into consideration that a factor $k^3/2\pi^2$ should appear in (3.33) in order to compute the variance of v.

We vary (3.36) with respect to k in order to quantify deviations from scale-invariance. This provides us the spectral index n_s and the running of the spectral index α_s

$$n_s - 1 = \left.\frac{d\ln\mathcal{P}_\zeta}{d\ln k}\right|_{k_*} , \tag{3.37}$$

$$\alpha_s = \left.\frac{d^2\ln\mathcal{P}_\zeta}{d(\ln k)^2}\right|_{k_*} . \tag{3.38}$$

In the slow roll regime, the inflaton field and comoving wavenumber are related by the scalar field equations of motion, which provide us

$$\frac{d}{d\ln k} = -\frac{V_\varphi}{3H^2}\frac{d}{d\varphi} , \tag{3.39}$$

where we used the fact that we are considering $k = aH$, and H is essentially constant, so $d\ln k = d\ln a = (H/\dot{\varphi})d\varphi$.

Then, the spectral index and its running can be expressed in terms of the potential and its derivatives

$$n_s - 1 = 2\eta_V - 6\epsilon_V , \tag{3.40}$$

$$\alpha_s = -2\xi_V - 24\epsilon_V^2 + 16\epsilon_V\eta_V . \tag{3.41}$$

Going further, we can apply the same procedure in order to calculate the power spectrum of tensor perturbations $\mathcal{P}_h$. Now, we will solve a equation in form of

$$h'' + \left(k^2 - \frac{a''}{a}\right) h = 0 , \tag{3.42}$$

instead of (3.27), and find

$$\mathcal{P}_h(k_*) = \frac{8}{M_{Pl}^2}\left(\frac{H}{2\pi}\right)^2 = \frac{2V}{3\pi^2 M_{Pl}^4} . \tag{3.43}$$

By comparing (3.36) and (3.43), it is clear to see that while the scalar amplitude depends on both H and ϵ, the tensor amplitude is a direct measure of the expansion rate H.

The tensor-to-scalar ratio r is calculated by

$$r = \frac{\mathcal{P}_h(k_*)}{\mathcal{P}_\zeta(k_*)} \backsimeq 16\epsilon_V \; . \tag{3.44}$$

Until now, tensor modes have not been observed, then we only have an upper limit on their amplitude $r < 0.064$.

There is a huge amount of inflationary models, but since the data seem to favour the single field inflation [15], by our analysis in this section, it is quite straightforward to compute the observables for them. Some inflationary models are shown in Fig.3.1. They are constrained in the parameter space n_s vs. r. It is clear that the R^2 inflation has a special spot in this plot, and this is why we dedicate the next section to the Starobinsky model [72].

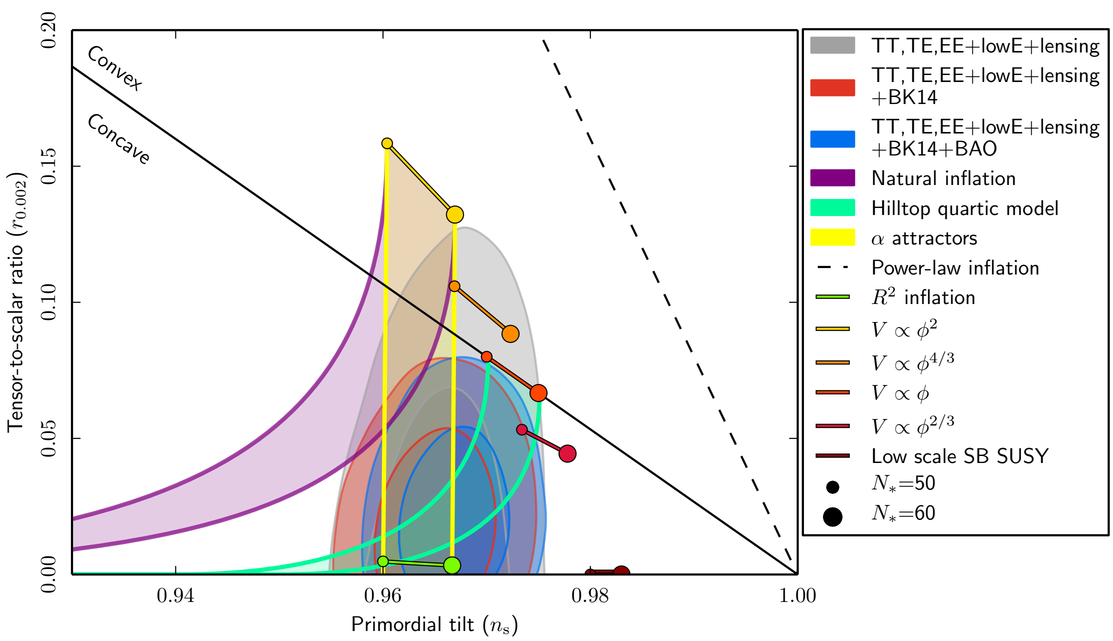


Fig. 3.1.: This figure shows the marginalized joint 68% and 95% CL regions for n_s and r at $k = 0.002\mathrm{Mpc}^{-1}$ from Planck alone and in combination with BK14 or BK14 plus BAO data, compared to the theoretical predictions of selected inflationary models. Image credit: Ref.[1].

4

$f(R)$ Theory, The Starobinsky Model and Inflation

The high accuracy of observations have put inflation in the precision era due to the fact that Planck experiments can now constrain rigorously inflationary models. The objective of this chapter is to present the Starobinsky model, its predictions and a possible modification to it inspired by the α-Attractors, since $f(R)$ models have been considered good candidates to compute the spectrum of primordial scalar perturbations generated in this inflationary stage. Also, we propose a mapping between the slow roll parameters and non-Gaussianities using a generic form of $f(R)$ in order to find an easy way to add these results as a pipeline over the constraining of the observational parameters. By generic form of $f(R)$ we mean that in principle we could obtain the slow roll parameters for any $f(R)$, since to apply our definitions we just need a $f(R)$ model and its derivatives in terms of R. On the other hand, we need to stress that slow roll inflation may only occur for the range of R in which $f(R)/R^2$ is a slowly changing function of R, namely, where its first and second derivatives with respect to $\ln R$ are small by modulus, as shown in [73].

The following sections are based on the results obtained in [41, 42].

Reconstructing the α-Attractors from an $f(R)$ theory

In this section, we describe briefly the $f(R)$ theory, with emphasis on the Starobinsky model. We also point out that there is an analogy at high curvature between the $f(R)$ theory and the α-Attractors. We will show that the power law correction R^{n-1} allows for a production of gravitational waves enhanced with respect to the one in the Starobinsky model, while maintaining a viable prediction on n_s. We numerically reconstruct the full α-

Attractors class of models testing the goodness of our high-energy approximation. Moreover, we also investigate the case of a single power law $f(R) = \gamma R^{2-\delta}$ theory, with γ and δ free parameters, to confirm once again the excellent agreement between the Starobinsky model and observation.

$f(R)$ theory

We start this section by briefly reviewing how an $f(R)$ theory is able to provide a viable inflationary model. Let's start with the following action

$$S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} f(R) \,, \tag{4.1}$$

where $M_{\rm Pl}$ is the Planck mass. Following [74, 75], the above $f(R)$ action (4.1) is equivalent to the Einstein-Hilbert one with a minimally coupled (to the conformally transformed metric) canonical scalar field χ, defined as follows

$$\chi = \sqrt{\frac{3}{2}} M_{\rm Pl} \ln\left(\frac{df}{dR}\right) \,, \tag{4.2}$$

and subject to the following potential

$$U = \frac{M_{\rm Pl}^2}{2(df/dR)^2} \left(R\frac{df}{dR} - f\right) \,. \tag{4.3}$$

At this point one can already see why the Starobinsky model $f(R) = R + \frac{R^2}{6M^2}$ is special. Assume for example a $R+\gamma R^n$ theory, with $n > 0$ and where γ is some energy scale at which the correction becomes relevant. At high energy scales $\gamma R^n \gg R$, where we expect Inflation to take place, the potential U becomes

$$U = \frac{M_{\rm Pl}^2(n-1)}{2\gamma n^2} R^{2-n} \,. \tag{4.4}$$

The Starobinsky case $n = 2$ then provides a plateau, which is the ideal situation for a slow roll inflationary phase to take place. If $n > 2$ the potential initially increases and then goes to zero asymptotically, which might be bad from the point of view of Inflation because the slow rolling scalar field would need to overcome a potential barrier. For $n < 2$ the potential grows unbound, but still there is the possibility for it to satisfy the slow roll conditions. See [76, 77, 78].

Dynamical system perspective

Let us analyse the above-described asymptotic behaviours from a dynamical system perspective. Assume a spatially flat FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j \ . \tag{4.5}$$

It is not difficult to cast the evolution equations of $f(R)$ gravity in absence of matter as the following dynamical system

$$\begin{cases} 6Hf''\dot{R} = Rf' - f - 6f'H^2 \\ 6\dot{H} = R - 12H^2 \end{cases} , \tag{4.6}$$

where $H \equiv \dot{a}/a$, the dot denotes derivation with respect to the cosmic time and the prime denotes derivation with respect to R. Note that the second equation of system (4.6) is the very definition of R for metric (4.5) and we have assumed f'' and H different from zero.

In order to have an inflationary phase for large R (larger than a certain scale M^2), we need $\dot{H} \approx 0$. From the second equation of system (4.6) this implies $R \approx 12H^2$ and thus $\dot{R} \approx 0$. This happens if, from the first equation of system (4.6), we obtain

$$\frac{f'}{f} \approx \frac{2}{R} , \qquad \text{or} \qquad \frac{Rf'/2 - f}{6Hf''} \sim 0 \ . \tag{4.7}$$

The first condition leads us to

$$f(R) \sim R^2 , \tag{4.8}$$

which is the Starobinsky R^2 correction to the Einstein-Hilbert action. The second condition tells us that $6Hf''$ must grow more rapidly than $Rf'/2 - f$ for large R. This can be achieved via a R^n correction with $n > 2$, but then the problem that we have mentioned earlier appears:

the scalar field has to climb a potential barrier in order for Inflation to end. Recently, an investigation of the inflationary dynamics generated by an inflaton field climbing up a potential has been investigated in [79, 80], where the authors present viable scenarios. Another recent interesting proposal is [81] where a logarithmic correction to the Einstein-Hilbert action is investigated. In this case, the divergence is linear since the R term dominates but the presence of the logarithmic correction produces a plateau at intermediate energies where Inflation might take place.

The potential (4.3) for the Starobinsky model can be computed exactly and has the following form

$$U(\chi) = \frac{3}{4} M^2 M_{\text{Pl}}^2 \left(1 - e^{-\sqrt{2/3}\chi/M_{\text{Pl}}}\right)^2 . \tag{4.9}$$

This potential presents a plateau for high values of the inflation field, as shown in Fig. 4.1.

Considering the excellent fit to the latest Planck data, in the next subsections we are going to focus on two equally interesting subclasses of the α-attractors: one in which the potential depends explicitly on $e^{-\sqrt{2/(3\alpha)}\chi/M_{\text{Pl}}}$, known as E-model, and another in which the potential depends explicitly on $\tanh^2\left(\frac{\chi}{\sqrt{6\alpha}M_{\text{Pl}}}\right)$, known as T-model.

The α-Attractors: E-models

The subclass of α-Attractors called E-Models [24] is described by the following scalar field potential

$$U_E(\chi) = \frac{3}{4} M^2 M_{\text{Pl}}^2 \alpha \left(1 - e^{-\sqrt{2/(3\alpha)}\chi/M_{\text{Pl}}}\right)^2 . \tag{4.10}$$

Clearly, the Starobinsky potential (4.9) is recovered for $\alpha = 1$. In Fig. 4.1 we display the behavior of the above potential for different choices of α.

Applying the definitions given in (4.2) and (4.3) to the E-Models potential of (4.10), it is not difficult to obtain the following differential equation for $f(R)$

$$Rf' - f = \frac{3M^2}{2(1-\beta)^2}\left(f' - f'^{\beta}\right)^2 , \tag{4.11}$$

where we have defined

$$\beta \equiv 1 - \frac{1}{\sqrt{\alpha}} \,. \tag{4.12}$$

Note that, since $\alpha > 0$, then $\beta < 1$. When $\alpha = 1$, i.e. $\beta = 0$, one can easily check that the Starobinsky model is solution of (4.11). When $\alpha \to \infty$, i.e. $\beta \to 1$, it is easy to see that from (4.10) we get a χ^2 potential and the corresponding $f(R)$ theory can be found by solving the equation

$$Rf' - f = \frac{3}{2} M^2 f'^2 \ln^2(f') \,. \tag{4.13}$$

Finally, note that $f = R$ turns (4.11) into a identity. Therefore, (4.11) can be seen as an equation which determines the correction to the usual Einstein-Hilbert action which is able to reproduce the inflationary dynamics of the α-Attractors E-models.

Unfortunately, (4.11) cannot be solved analytically for a generic α and we analyse it numerically in the AppendixA. On the other hand, if there exists a $f(R)$ theory reconstructing the same inflationary dynamics as the α-Attractors, then at high energies the former must behave as the Starobinsky model, i.e. $f(R) \propto R^2$, because the α-Attractors potential displays a plateau at high energies and, in the framework of $f(R)$ theories, this is realized only by a $f(R) \propto R^2$ correction.

Let us show this explicitly, assuming a $f(R) = \gamma R^n$ theory, where $\gamma > 0$ is an arbitrary energy scale and $n > 1$, and substituting it into (4.11)

$$\frac{2(n-1)(1-\beta)^2}{3M^2\gamma n^2} R^{2-n} = \left[1 - (\gamma n)^{\beta-1} R^{(\beta-1)(n-1)}\right]^2 \,. \tag{4.14}$$

Since $\beta < 1$ and $n > 1$, the exponent $(\beta - 1)(n - 1)$ is negative. Therefore, in the limit $\gamma R^{n-1} \to \infty$, we get from (4.14)

$$\frac{2(n-1)(1-\beta)^2}{3M^2\gamma n^2} R^{2-n} \to 1 \,. \tag{4.15}$$

In order to satisfy this limit, n must be equal to 2. This result confirms that the asymptotic behavior of a $f(R)$ theory which aims to reproduce the α-Attractors dynamics must go as R^2

for large R. We then propose this *ansatz*

$$f(R) = R + aR^n + bR^2 \ , \tag{4.16}$$

where $a > 0$ and $b > 0$ are arbitrary energy scales and $n > 1$. Let's substitute (4.16) into (4.11)

$$\frac{a(n-1)R^n + bR^2}{(1 + anR^{n-1} + 2bR)^2} = \frac{3M^2}{2(1-\beta)^2}\left[1 - (1 + anR^{n-1} + 2bR)^{\beta-1}\right]^2 \ , \tag{4.17}$$

and consider the high energy limits

$$aR^{n-1} \gg 1 \ , \quad \text{and} \quad bR \gg 1 \ . \tag{4.18}$$

If $n < 2$, the dominant contributions in (4.17) are those proportional to R^2 and we get from them

$$b = \frac{(1-\beta)^2}{6M^2} \ . \tag{4.19}$$

For $\beta = 0$ we then recover the correct form of the coefficient for the Starobinsky model. One can check that in (4.17) the next-to-leading power in n, being $n < 2$, is n itself, whereas the next-to-leading power in β, being $\beta < 1$, is $\beta + 1$. Therefore, we can approximate (4.17) as follows

$$a(n-1)R^n + bR^2 \sim \frac{3M^2}{2(1-\beta)^2}\left[4b^2R^2 + a^2n^2R^{2(n-1)} + 4banR^n - 2(2bR)^{\beta+1}\right] \ . \tag{4.20}$$

We equate the sub-dominant powers and their respective coefficients, obtaining

$$n = \beta + 1 \qquad a = (2b)^\beta \ . \tag{4.21}$$

When $\beta = 0$, the aR^n contribution of (4.16) is thus reincorporated into the usual Einstein-Hilbert term of the action and we are left with the Starobinsky model. The comparison between powers that we have just done in (4.17) allows us to reconstruct the α-Attractors at high energies as a polynomial correction to the Starobinsky model (4.16)

$$f(R) = R + \left[\frac{(1-\beta)^2}{3M^2}\right]^\beta R^{\beta+1} + \frac{(1-\beta)^2}{6M^2}R^2 \ . \tag{4.22}$$

This is just an approximation because the other sub-leading powers do not compensate, being indeed (4.16) not a solution of (4.11).

If $n \geq 2$ in (4.17), one can easily check that the equation can be balanced only if $n = 2$. This is expected because, as we have already commented earlier, in order to provide a scalar potential with a plateau at high energies, a $f(R)$ theory must go asymptotically as R^2.

The α-Attractors: T-models

There exists another subclass of the α-Attractors named T-Models. The potential characterising these models is the following

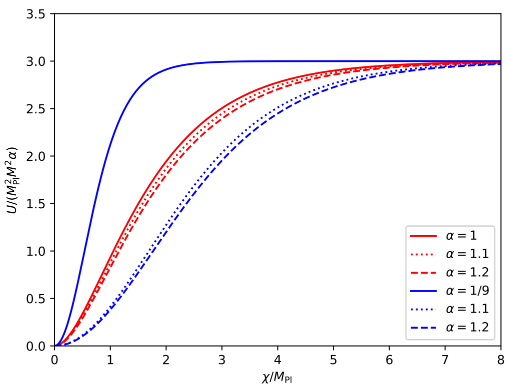
$$U_T(\chi) = 3M^2 M_{\mathrm{Pl}}^2 \alpha \tanh^2 \left(\frac{\chi}{\sqrt{6\alpha} M_{\mathrm{Pl}}} \right) , \tag{4.23}$$

i.e. based on the plateau-like behavior of the hyperbolic tangent for large values of the field. Indeed, the above potential recovers the E-models potential, and therefore the Starobinsky one when $\alpha = 1$, only for large values of the field. In Fig. 4.1 we display the evolution of the E-models and T-models for some values of α including $\alpha = 1$, i.e. the Starobinsky model.

In the large α limit, by performing a Taylor-series expansion, it is easy to see that the T-models potential (4.23) coincides with the one of the chaotic inflation model with a quadratic potential [22], as for the E-models.

In the large α limit, by performing a Taylor-series expansion, it is easy to see that the T-models potential (4.23) coincides with the one of the chaotic inflation model with a quadratic potential [22], as for the E-models.

As we did in the previous subsection, using the definitions given in (4.2) and (4.3) for the T-Model potential of (4.23), we can obtain a differential equation which allows to



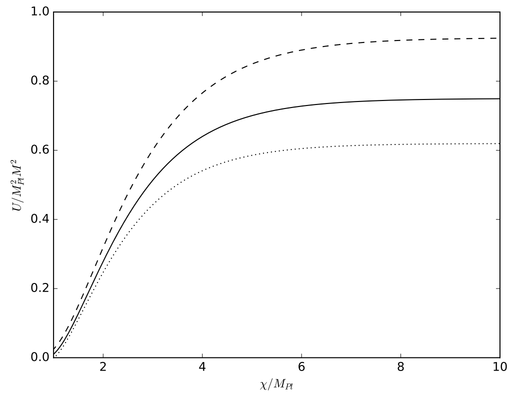


Fig. 4.1.: *Left Panel.* Evolution of the E-Models potentials (4.10) (red lines) and of the T-models potentials (4.23) (blue lines) for some values of α. The choices $\alpha = 1$ and $\alpha = 1/9$ are interesting because they reproduce the Starobinsky model and the chaotic inflation model [2, 3].
Right Panel. Evolution of the normalised potential derived from Eq. (4.30). The solid line is drawn for the value $\beta = 1$, i.e. the Starobinsky model. The dashed and the dotted line are for $\beta = 0.1$ and $\beta = -0.1$, respectively.

reconstruct a corresponding $f(R)$

$$f'^2 \frac{\left[1 - f'^{(\beta-1)}\right]^2}{\left[1 + f'^{(\beta-1)}\right]^2} = \frac{(1-\beta)^2}{6M^2}(Rf' - f)\ . \tag{4.24}$$

In the limit of large fields, i.e. $f'^{\beta-1} \to 0$,[1] the above equation becomes similar to (4.11). Therefore, the same asymptotic analysis which led us to justify the *ansatz* given in (4.22) applies.

A power law extension of the Starobinsky model

Motivated by the result of the previous subsection, here we address in detail the model given by (4.22), i.e. a power law extension of the Starobinsky model. In this case it is not possible to find a closed, analytic form for the potential of (4.3) as function of the field χ of (4.2). In Refs. [82, 83] the authors show that the following potential (written here in our notation)

$$U(\chi) = M_{\rm Pl}^2 \left(\alpha_1 - \gamma_1 e^{-n\sqrt{2/3}\chi/M_{\rm Pl}}\right)\ , \tag{4.25}$$

[1] This limit holds true only if f' is a growing function of R, i.e. $f'' > 0$, which is a condition that we are assuming.

where α_1 and γ_1 are parameter with dimension of a square mass, is able to reproduce asymptotically the following $f(R)$ model

$$f(R) \sim \frac{R^2}{4\gamma_1(2-n)} + \frac{1}{2-n}\left[\frac{1}{2\gamma_1(2-n)}\right]^{1-n} R^{2-n}\,, \tag{4.26}$$

via an analysis similar to the one performed in the previous section. From the above potential, it is possible to derive the following predictions on the scalar index and the tensor-to-scalar ratio

$$n_s \simeq 1 - \frac{2}{N}\,, \qquad r \simeq \frac{12}{n^2 N^2}\,, \tag{4.27}$$

that is, a behaviour similar to the one predicted by the α-Attractors.

Our analysis of the model (4.22) shall be purely numerical. The derivative of (4.22), is the following

$$f'(R) = 1 + (1-\beta)^{2\beta}(1+\beta)\left(\frac{R}{3M^2}\right)^{\beta} + (1-\beta)^2\frac{R}{3M^2}\,. \tag{4.28}$$

This expression suggests the use of the dimensionless variable $R/3M^2$. Indeed, one can easily show that also the potential (4.3) is a function of $R/3M^2$, since

$$U = \frac{3M^2 M_{\rm Pl}^2}{2(f')^2}\left(\frac{R}{3M^2}f' - \frac{f}{3M^2}\right)\,, \tag{4.29}$$

and we have just shown that f' is function of $R/3M^2$ and

$$\frac{f}{3M^2} = \frac{R}{3M^2} + (1-\beta)^{2\beta}\left(\frac{R}{3M^2}\right)^{\beta+1} + \frac{(1-\beta)^2}{2}\left(\frac{R}{3M^2}\right)^2\,, \tag{4.30}$$

is evidently function of $R/3M^2$. In Fig. 4.1 we display the evolution of the potential corresponding to (4.22).

The slow roll parameters can be defined as follows, given the functional form of the potential $U(\chi)$

$$\epsilon_U \equiv \frac{M_{Pl}^2}{2}\left(\frac{U_\chi}{U}\right)^2\,, \quad \eta_U \equiv M_{Pl}^2\frac{U_{\chi\chi}}{U}\,, \quad \xi_U \equiv M_{\rm Pl}^4\frac{U'U'''}{U^2}\,, \quad \sigma_U \equiv M_{\rm Pl}^6\frac{U'^2U^{(4)}}{U^3}\,, \tag{4.31}$$

where $U^{(4)}$ represents the fourth derivative of the potential with respect to the field. From these quantities it is straightforward to compute the scalar index and the tensor-to-scalar ratio

$$n_s \equiv 1 - 6\epsilon_U + 2\eta_U \ , \quad r \equiv 16\epsilon_U \ , \tag{4.32}$$

and the running and the running of the running of the scalar spectral index

$$\alpha_s \equiv \frac{dn_s}{d\log k} = -2\xi_U^2 + 16\eta_U\epsilon_U - 24\epsilon_U^2 \ , \tag{4.33}$$

$$\beta_s \equiv \frac{d\alpha_s}{d\log k} = 2\sigma_U^3 + 2\xi_U^2(\eta_U - 12\epsilon_U) - 32\epsilon_U(\eta_U^2 - 6\eta_U\epsilon_U + 6\epsilon_U^2) \ . \tag{4.34}$$

For single field inflationary models, the above runnings are of the order of $\alpha_s \sim 10^{-3}$ and $\beta_s \sim 10^{-5}$ [84, 85]. Discriminating among α_s and β_s of different models is still an experimental challenge, however it might be possible to use α_s to this purpose in forthcoming Stage-4 CMB experiments, see e.g. [86]. In Fig. 4.2 we display the evolution of n_s and r as functions of β, showing a good agreement with the observational constraints for a range $0 < \beta \lesssim 0.8$. The latter value corresponds to a $R^{1.8}$ correction to the Starobinsky model. In Fig. 4.2 we also plot the runnings as functions of β and, finally, in Fig. 4.3 we display the prediction of the model investigated in this section on the r *vs* n_s plane by varying β in the interval $-0.02 < \beta < 0.8$ (corresponding to $0.96 < \alpha < 25$) and comparing this evolution with the contour regions allowed from the Planck data [4]. As one can appreciate, the polynomial correction to R^2 allows larger values of r. In particular, it seems that diminishing the number of e-folds could allow to even larger values of r.

Study of the single power law $f(R)$ inflationary model

In this subsection we address, for completeness, the case of a single power-law $f(R)$

$$f(R) = R + \frac{R^{2-\delta}}{(6M^2)^{1-\delta}} \ , \tag{4.35}$$

and compute its predictions on the inflationary parameters. This kind of model has already been investigated in [87], in order to test the robustness of the Starobinsky model and assess

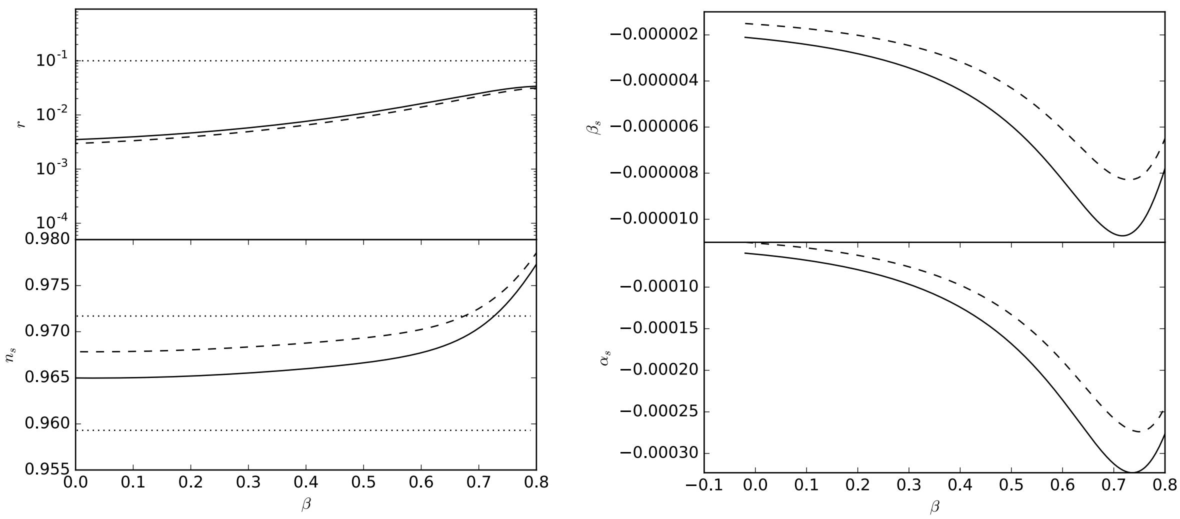


Fig. 4.2.: *Left Panel.* Evolution of the inflationary parameters n_s and r as functions of β derived from (4.30). The solid line is drawn for $N = 55$ whereas the dashed line represents the case $N = 60$. The dotted lines represent the observational constraints at 95% CL.
Right Panel. Evolution of the runnings as functions of β derived from (4.30). The solid line is drawn for $N = 55$ whereas the dashed line represents the case $N = 60$.

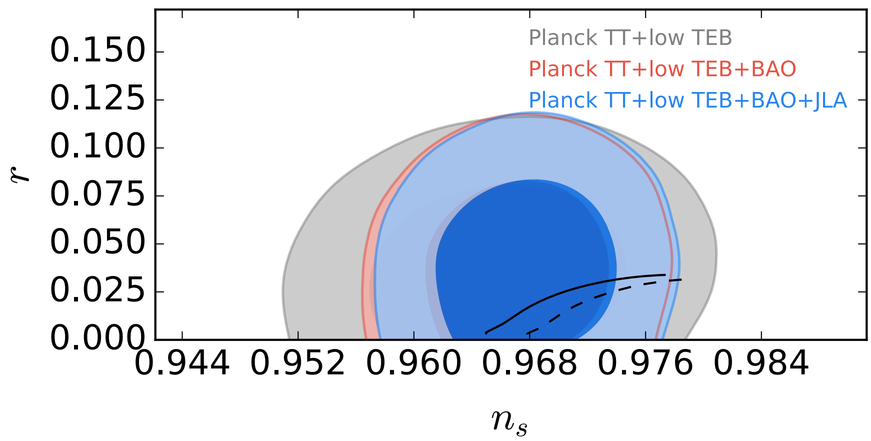


Fig. 4.3.: Evolution of r *vs* n_s, varying β in the interval $-0.02 < \beta < 0.8$ (corresponding to $0.96 < \alpha < 25$) and for $N = 55$ (solid line) and $N = 60$ (dashed line) with the marginalized 68% and 95% confidence level contours from Planck 2015 data [4].

how much precise data have to be in order to detect deviations from the case $\delta = 0$. We present a similar analysis in next section, providing analytic results for the inflationary parameters n_s and r as functions of the number of e-folds N and the new parameter δ.

Starting from (4.35), we can determine the following potential for the scalar field χ

$$U(\chi) = \frac{3M_{\rm Pl}^2 M^2 (1-\delta)}{(2-\delta)^{\frac{2-\delta}{1-\delta}}} e^{-2\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}} \left(e^{\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}} - 1 \right)^{\frac{2-\delta}{1-\delta}} . \tag{4.36}$$

It is straightforward to see that for $\delta = 0$ we recover the Starobinsky model given in (4.9). For large fields, the above potential behaves as

$$U(\chi) \sim \frac{3M_{\rm Pl}^2 M^2 (1-\delta)}{(2-\delta)^{\frac{2-\delta}{1-\delta}}} e^{\sqrt{\frac{2}{3}}\frac{\delta}{1-\delta}\frac{\chi}{M_{\rm Pl}}} , \qquad (\chi \gg M_{\rm Pl}) , \tag{4.37}$$

where again, for $\delta = 0$, we recover the plateau typical of the Starobinsky model. In Fig. 4.4 we show the behaviour of $U(\chi)$ of the model (4.36) for different values of δ and of (4.10) for different values of α, compared with the Starobinsky model.

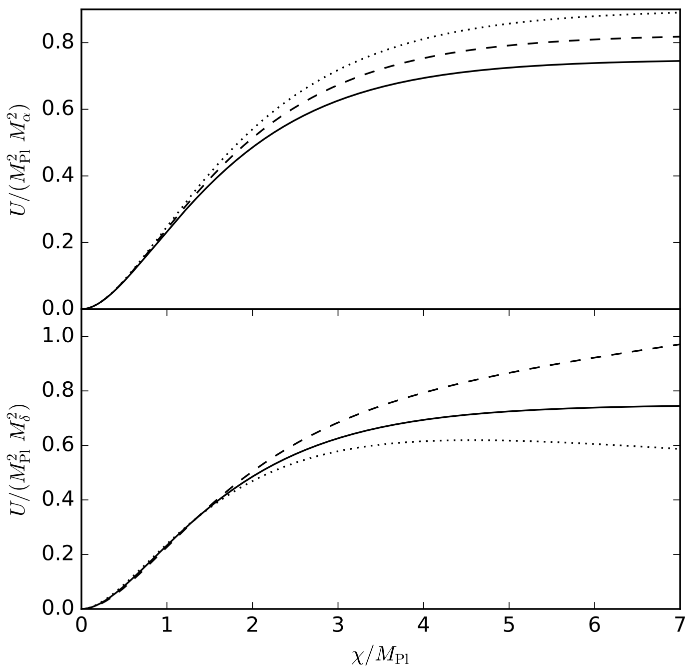


Fig. 4.4.: Upper panel. Evolution of the potential (4.10) for $\alpha = 1.1$ (dashed line) and $\alpha = 1.2$ (dotted line) compared with the Starobinsky case (solid line). Lower panel. Evolution of the potential (4.36) for $\delta = 0.05$ (upper dashed line) and $\delta = -0.05$ (lower dotted line) compared with the Starobinsky case (solid line).

As expected, from Fig. 4.4 one can see that the α-Attractors potential seems to preserve the plateau at high energies, whereas the δ-model displays an important change in the steepness of the potential. This statement can be made more quantitative by calculating the slow roll parameters from (4.36). We have for ϵ_U

$$\epsilon_U \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{U_\chi}{U}\right)^2 = \frac{1}{3}\left(\frac{1}{1-\delta}\right)^2\left[\frac{\delta + 2(1-\delta)e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}}{1-e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}}\right]^2 , \tag{4.38}$$

and η_U:

$$\eta_U \equiv M_{\rm Pl}^2\frac{U_{\chi\chi}}{U} = \frac{2}{3(1-\delta)^2}\frac{\delta^2-(1-\delta)(2-5\delta)e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}+4(1-\delta)^2e^{-2\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}}{\left(1-e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}\right)^2} \tag{4.39}$$

For large fields ($\chi \gg M_{\rm Pl}$) the slow roll parameters can be approximated as follows

$$\epsilon_U \sim \frac{1}{3}\left(\frac{1}{1-\delta}\right)^2\left[\delta+(2-\delta)e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}\right]^2 , \tag{4.40}$$

and

$$\eta_U \sim \frac{2}{3(1-\delta)^2}\left[\delta^2-(2-7\delta+3\delta^2)e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}\right] . \tag{4.41}$$

Note that

$$\epsilon_U \to \frac{1}{3}\left(\frac{\delta}{1-\delta}\right)^2 , \qquad \chi\to\infty , \tag{4.42}$$

i.e. there is no plateau if $\delta > 0$, as we expected from observing Fig. 4.4 and from the analysis performed earlier in Sec. 4.1.1. Moreover, since $\epsilon_U \ll 1$ in order to have an inflationary phase, then $\delta \ll 1$.

The value of the field at which inflation ends is given by $\epsilon_U(\chi_f) \approx 1$, which in our case translates to

$$\epsilon_U(\chi_f)\approx 1 \quad \Rightarrow \quad e^{-\sqrt{\frac{2}{3}}\frac{\chi_f}{M_{\rm Pl}}} \approx \frac{\sqrt{3}-\delta(1+\sqrt{3})}{(2+\sqrt{3})(1-\delta)} \simeq \frac{\sqrt{3}-\delta}{2+\sqrt{3}} . \tag{4.43}$$

With this we can calculate the number of e-folds as follows

$$N = \frac{1}{\sqrt{2}M_{\rm Pl}} \int_{\chi_f}^{\chi_i} \frac{d\chi}{\sqrt{\epsilon_U}} \,. \tag{4.44}$$

Let's stay at the lowest possible order in both δ and $e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}$ and their combinations. In this case, we can approximate the slow roll parameters and the number of e-folds as follows

$$\epsilon_U \sim \frac{1}{3}\left(\delta + 2e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}}\right)^2 \,, \quad \eta_U \sim -\frac{4}{3}e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}} \,, \quad \frac{1}{N} \sim \frac{4}{3}e^{-\sqrt{\frac{2}{3}}\frac{\chi}{M_{\rm Pl}}} + \frac{\delta}{3} \,. \tag{4.45}$$

The scalar index and the tensor-to-scalar ratio are therefore written as

$$n_{\rm s} \equiv 1 - 6\epsilon_U + 2\eta_U \sim 1 - \frac{2}{N}\left(1 - \frac{N\delta}{3}\right) \,, \tag{4.46}$$

$$r \equiv 16\epsilon_U \sim \frac{12}{N^2}\left(1 + \frac{N\delta}{3}\right)^2 \,. \tag{4.47}$$

As we can see, differently from the α-Attractors case, the δ correction interferes also with the scalar spectral index, through the combination $N\delta$. In Fig. 4.5 we display the numerical results for the evolution of $n_{\rm s}$ and r as functions of δ, by fixing the number of e-folds $N = 55$ and $N = 60$. For completeness, in the same figure we also calculate the nonvanishing runnings $\alpha_{\rm s} = dn_{\rm s}/d\log k$ and $\beta_{\rm s} = d\alpha_{\rm s}/d\log k$.

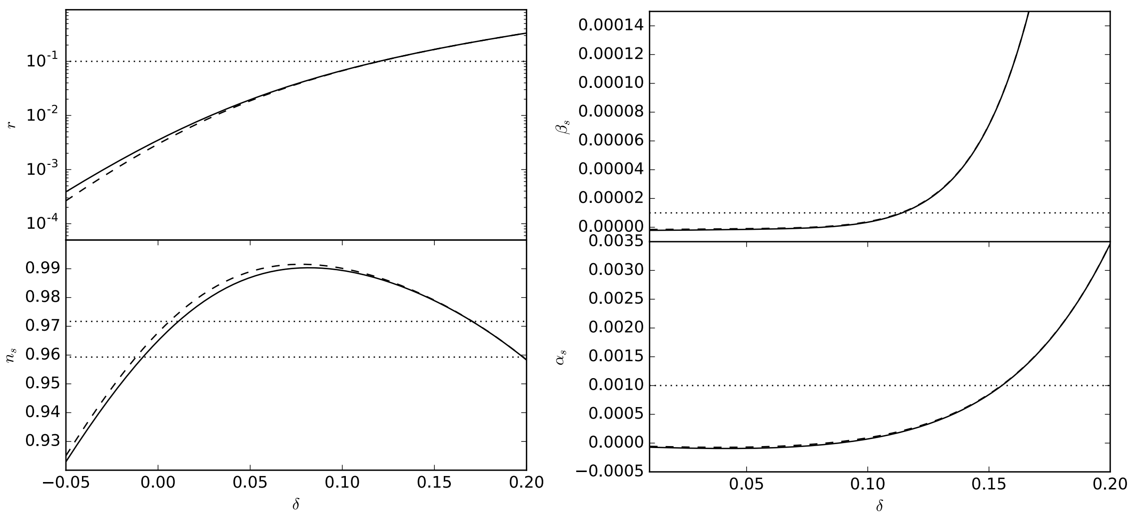


Fig. 4.5.: Evolution of r, $n_{\rm s}$, $\beta_{\rm s}$ and $\alpha_{\rm s}$ as functions of δ for $N = 55$ (solid line) and $N = 60$ (dashed line). The horizontal dotted line is the upper limit on r, at 95% confidence level, obtained by the Planck collaboration [4]. The horizontal dotted lines enclose the 68% confidence level of the values of $n_{\rm s}$ measured by the Planck collaboration [4] .

In Fig. 4.6 we display the δ model in the r *vs* n_s plane, again choosing $N = 55$ and $N = 60$. When $N = 55$, the variation of the δ parameter is $-0.004 < \delta < 0.011$ for 1σ, and $-0.011 < \delta < 0.022$ for 2σ. When $N = 60$, it is $-0.008 < \delta < 0.006$ for 1σ, and $-0.015 < \delta < 0.016$ for 2σ. This figure is very similar to the first panel of Fig. 4 of Ref. [87].

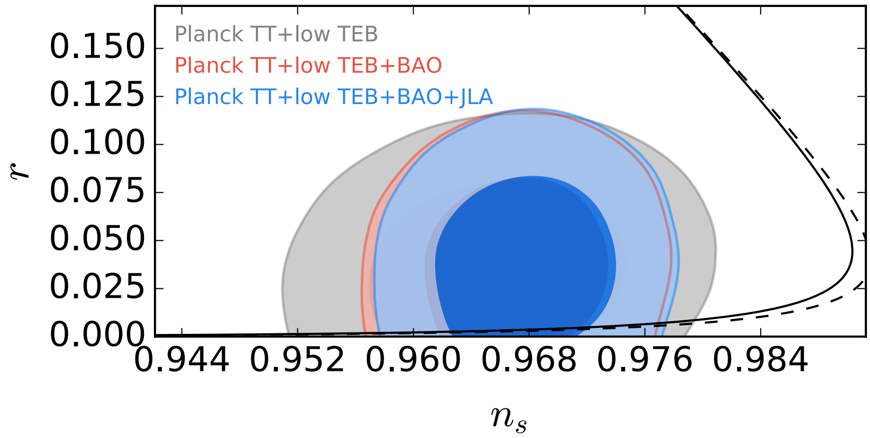


Fig. 4.6.: Evolution of r *vs* n_s, varying δ and for $N = 55$ (solid line) and $N = 60$ (dashed line) with the marginalized 68% and 95% confidence level contours from Planck 2015 data.

Differently from the α-Attractors case, the δ model move in the "wrong" direction in the r *vs* n_s, i.e. they move mostly horizontally.

Generic slow roll and non-gaussianity parameters in $f(R)$ theories

In this section we establish *formulae* for the inflationary slow roll parameters ϵ, η and ζ as functions of the Ricci scalar R for $f(R)$ theories of gravity. As examples, we present the analytic and numerical solutions of ϵ, η and ζ as functions of the number of e-folds N in two important instances: for the Starobinsky model and for the $f(R)$ reconstruction of the α-Attractors (4.16). The highlight of our proposal is to rewrite the slow roll parameters in terms of $f(R)$, which allows to find directly inflationary parameters as functions of R itself.

Generic slow roll parameters for a $f(R)$ model

As shown in Sec.4.1.1, a generic $f(R)$ theory can be mapped via a conformal transformation into General Relativity with a matter content constituted by a scalar field χ. In this context, we can use the slow roll definitions given by (4.31), where

$$U_\chi \equiv \frac{dU}{d\chi} = \frac{dU/dR}{d\chi/dR}\,, \tag{4.48}$$

in order to find:

$$\epsilon = \frac{1}{3}\left(\frac{2f - Rf'}{Rf' - f}\right)^2 . \tag{4.49}$$

$$\eta = \frac{2(f')^2}{3f''(Rf' - f)} - \frac{2Rf'}{Rf' - f} + \frac{8}{3}\,, \tag{4.50}$$

$$\zeta = \frac{4}{9}\frac{(2f - Rf')}{(Rf' - f)^2}\left[-\frac{f'''(f')^3}{(f'')^3} - \frac{3(f')^2}{f''} + 8f - Rf'\right]. \tag{4.51}$$

Applying these generic formulae to the Starobinsky model $f = R + \alpha R^2$, we can easily compute

$$\epsilon = \frac{1}{3\alpha^2 R^2}\,, \tag{4.52}$$

$$\eta = \frac{1 - 2\alpha R}{3\alpha^2 R^2}\,, \tag{4.53}$$

$$\zeta = \frac{2(2\alpha R - 3)}{9\alpha^3 R^3}\,. \tag{4.54}$$

It is straightforward to notice that the above slow-roll parameters tend to zero for $\alpha R \to \infty$, as we expect from the fact that the potential presents a plateau for the Starobinsky model for $\alpha R \to \infty$.

It is remarkable that *we are able to express the slow roll parameters as functions of R for any $f(R)$ theory*. However, in order to make contact with observation, we need to relate R to the number of e-folds N from the end of inflation. A simple way to compute N for a generic $f(R)$ theory in the slow roll approximation is rewriting (4.44) as

$$\begin{aligned} N &\approx \frac{1}{M_{\rm Pl}} \int_{\chi_1}^{\chi_2} d\chi \frac{1}{\sqrt{2\epsilon}} \\ &= \frac{3}{2} \int_{R_f}^{R_i} dR \left(\frac{f''}{f'} \right) \frac{Rf' - f}{2f - Rf'} \,, \end{aligned} \tag{4.55}$$

where the subscripts i and f over the integral denote an initial arbitrary moment during the inflationary phase and the final period of inflation, given by the condition $\epsilon_f \approx 1$.

In order to perform the integration in (4.55), we need an explicit form for our $f(R)$ theory. Again, for the Starobinsky model we get

$$N \approx 3 \int_{R_f}^{R_i} dR \frac{\alpha^2 R}{1 + 2\alpha R} \sim \frac{3}{2} \alpha (R_i - R_f) \,, \tag{4.56}$$

where we are assuming $\alpha R \gg 1$, which is a necessary condition to have an inflation stage. Considering $R_i \gg R_f$ and dropping the subscript i, we can write

$$\alpha R \sim \frac{2N}{3} \,. \tag{4.57}$$

Now we are able to write the slow roll parameters as

$$\epsilon = \frac{3}{4N^2} \,, \qquad \eta = -\frac{1}{N} \,, \qquad \zeta = \frac{1}{N^2} \,, \tag{4.58}$$

and the tensor-to-scalar ratio and scalar spectral index as

$$r \equiv 16\epsilon = \frac{12}{N^2} \,, \qquad n_{\rm s} \equiv 2\eta + 1 = 1 - \frac{2}{N} \,, \tag{4.59}$$

which for $N = 50 - 60$ give predictions in very good agreement with observations.

The running of the spectral index can be written as

$$\alpha_{\rm s} \equiv \frac{dn_{\rm s}}{d\ln k} = -2\zeta + 16\eta\epsilon - 24\epsilon^2 = -\frac{2}{N^2} \,, \tag{4.60}$$

which it is of the same order of the tensor-to-scalar ratio.

As a second case, we consider (4.22), which is a $f(R)$ approximated reconstruction of the α-Attractors, rewritten as

$$f(R) = R + \sigma^{\beta} R^{\beta+1} + \frac{\sigma}{2} R^2, \tag{4.61}$$

where $\sigma = (1-\beta)^2/3M^2$. We notice that when $\beta = 0$ we recover the Starobinsky model modified by a constant in the linear term and when $\beta = 1$ we recover GR.

Using (4.49)-(4.50)-(4.51) we can compute the following slow roll parameters and the number of e-folds for this model

$$\epsilon = \frac{4}{3}\left[\frac{1+(1-\beta)(\sigma R)^{\beta}}{2\beta(\sigma R)^{\beta}+\sigma R}\right]^2, \tag{4.62}$$

$$\begin{aligned}\eta = {} & \frac{8}{3} - 4\left[\frac{1+(\beta+1)(\sigma R)^{\beta}+\sigma R}{2\beta(\sigma R)^{\beta}+\sigma R}\right] \\ & + \frac{4}{3}\left\{\frac{1+2(\beta+1)(\sigma R)^{\beta}[1+\sigma R]+(\beta+1)^2(\sigma R)^{2\beta}+2\sigma R+(\sigma R)^2}{\sigma R\left[\beta(\beta+3)(\sigma R)^{\beta}+\sigma R+2\beta^2(\beta+1)(\sigma R)^{2\beta-1}\right]}\right\},\end{aligned} \tag{4.63}$$

$$\begin{aligned}\zeta = {} & \frac{16}{9}\frac{\left[1+(1-\beta)(\sigma R)^{\beta}\right]}{\left[2\beta(\sigma R)^{\beta}+\sigma R\right]^2}\left\{-\frac{\beta(\beta+1)(\beta-1)\left[1+(\beta+1)(\sigma R)^{\beta}+\sigma R\right]^3(\sigma R)^{\beta}}{(\sigma R)^3\left[1+\beta(\beta+1)(\sigma R)^{\beta-1}\right]^3}\right. \\ & \left. -3\frac{\left[1+(\beta+1)(\sigma R)^{\beta}+\sigma R\right]^2}{\sigma R\left[1+\beta(\beta+1)(\sigma R)^{\beta-1}\right]}+7+(7-\beta)(\sigma R)^{\beta}+3\sigma R\right\},\end{aligned} \tag{4.64}$$

$$N = \frac{3}{4}\int d(\sigma R)\left[\frac{2\beta^2(\beta+1)(\sigma R)^{2\beta-1}+\beta(\beta+3)(\sigma R)^{\beta}+\sigma R}{1+2(\sigma R)^{\beta}+(\beta+1)(1-\beta)(\sigma R)^{2\beta}+(1-\beta)(\sigma R)^{\beta+1}+\sigma R}\right]. \tag{4.65}$$

As it was shown in Sec.4.1.5, the viable range for β, determined from the Planck 2015 constraints on n_s and r, is $0 < \beta < 0.8$.

As in the Starobinsky case, expressed by (4.56), we assume that $\sigma R \gg 1$. Therefore, (4.65) can be approximated to

$$N \approx \frac{3}{4}\int_{\sigma R_f}^{\sigma R_i} d(\sigma R)\frac{\sigma R}{(1-\beta)(\sigma R)^{\beta+1}} \sim \frac{3}{4}\frac{(\sigma R)^{1-\beta}}{(1-\beta)^2}. \tag{4.66}$$

The latter allows us to express ϵ, η and ζ in terms of N as

$$\epsilon = \frac{3}{4(1-\beta)^2 N^2} , \tag{4.67}$$

$$\eta = -\frac{1}{N} , \tag{4.68}$$

$$\zeta = \frac{1}{(1-\beta)^2 N^2} . \tag{4.69}$$

We choose $\beta = 1/2$ and $N = 60$ as an explicit example in order to compare its predictions for the slow roll parameters, and consequently the non-Gaussianity function, with those of the Starobinsky model. For this case the slow roll parameters of the β-model are estimated as

$$\epsilon_{(\beta=1/2)} = 8.3 \times 10^{-4} , \tag{4.70}$$

$$\eta_{(\beta=1/2)} = -1.6 \times 10^{-2} , \tag{4.71}$$

$$\zeta_{(\beta=1/2)} = 1.1 \times 10^{-3} , \tag{4.72}$$

from which we can compute the tensor-to-scalar ratio, scalar spectral index and running of the scalar index as

$$r = 0.013 , \quad n_s = 0.966 , \quad \alpha_s = -0.002 . \tag{4.73}$$

We show the evolution of the slow roll parameters ϵ and ζ for the Starobinsky model and the β-model, with $\beta = 1/2$, as functions of N in Figure 4.7. The evolution of the tensor-to-scalar ratio versus the scalar spectral index for the Starobinsky model and for the β-model, with $\beta = 1/2$, are shown in Figure 4.8. For completeness, we express α_s for the Starobinsky model and the β-model, with $\beta = 1/2$ versus N and the evolution of α_s *vs* β are presented in Figure 4.9.

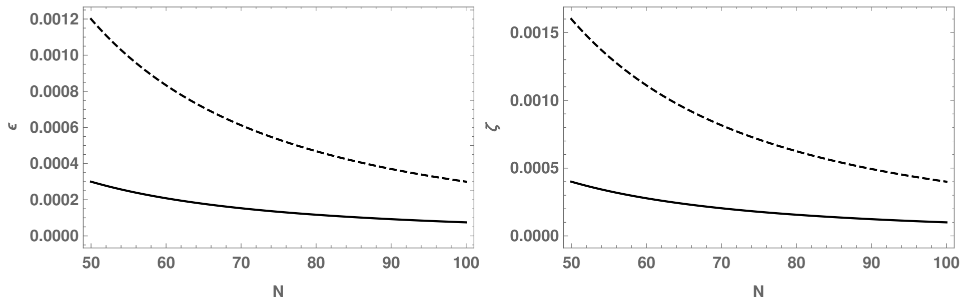


Fig. 4.7.: *Left Panel.* Evolution of ϵ as function of N for the Starobinsky model (solid line) and the β-model with $\beta = 1/2$ (dashed line). *Right Panel.* Evolution of ζ as function of N for the Starobinsky model (solid line) and the β-model with $\beta = 1/2$ (dashed line).

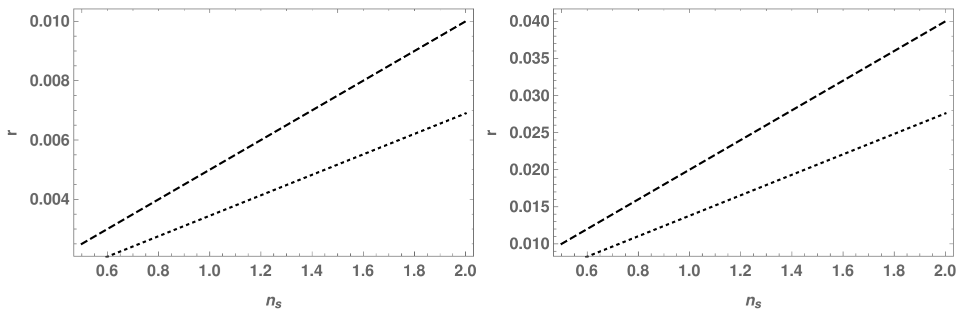


Fig. 4.8.: *Left Panel.* Prediction of the Starobinsky model for $N = 50$ (dashed line) and $N = 60$ (dotted line) on the space (n_s, r). *Right Panel.* Prediction of the β-model for $N = 50$ (dashed line) and $N = 60$ (dotted line) on the space (n_s, r).

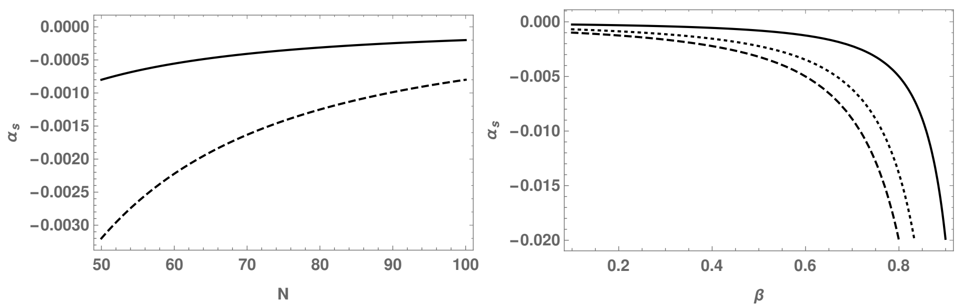


Fig. 4.9.: *Left Panel.* Evolution of α_s as function of N for the Starobinsky model (solid line) and the β-model with $\beta = 1/2$ (dashed line). *Right Panel.* Running of the spectral index for the β-model showing the cases : $N = 50$ (dashed line), $N = 60$ (dotted line) and $N = 100$ (solid line).

Generic non-gaussianities for a $f(R)$ model

The slow roll parameters defined in (4.31) from the potential U are related to the ones defined from the Hubble parameter as follows

$$\begin{aligned} \epsilon &\approx \epsilon_H \, , && (4.74)\\ \eta &\approx \epsilon_H + \eta_H , && (4.75)\\ \zeta &\approx \zeta_H + 2\epsilon_H(3\eta_H - \epsilon_H), && (4.76) \end{aligned}$$

Notice that for a successful inflation, we need $\epsilon \ll 1$, $\eta \ll 1$ for a standard Starobinksy model and additionally $\zeta \ll 1$ for an α-attractor model. Indirectly, large values of η are likely to make ϵ and ζ grow as well.

Another important tool to understand the primordial Universe is the non-gaussianity. Constraints on its observation can provide information about the nature of the inflaton, whether it is a canonical field or multiple fields, or if inflation happens in a higher order scalar tensor theories, and other possibilities, since every model produces a distinct signal for f_{NL} (depending on the configuration of triangles) of 3-momenta.

The standard, single-field slow-roll models of inflation predict adiabatic and gaussian primordial perturbations with a nearly scale-invariant spectrum, and their non-gaussianity parameter is given by [88]

$$f_{\mathrm{NL}} \sim (n_s - 1) \sim 10^{-2} \, , \tag{4.77}$$

which is consistent to the current Planck constraints $|f_{\mathrm{NL}}| \lesssim \mathcal{O}(10)$.

More explicitly, in standard inflation driven by a potential U_χ, we have the equilateral non-linear parameter as [89]

$$f_{\mathrm{NL}}^{\mathrm{equil}} = \frac{55}{36}\epsilon_H + \frac{5}{12}\eta_H \, , \tag{4.78}$$

therefore, using our definitions for the slow roll parameters (4.74), we can have a *generic non-Gaussianity parameter for $f(R)$*

$$f_{\mathrm{NL}}^{\mathrm{equil}} = \frac{30}{27}\epsilon + \frac{5}{12}\eta \,. \tag{4.79}$$

From this *generic* result, we can express the non-Gaussianity parameter in terms of N for the Starobinsky model as

$$f_{\mathrm{NL}}^{\mathrm{equil}} = \frac{15}{18N^2} - \frac{5}{12N} \,. \tag{4.80}$$

and for the β-model

$$f_{\mathrm{NL}}^{\mathrm{equil}} = \frac{15}{18(1-\beta)^2N^2} - \frac{5}{12N} \,. \tag{4.81}$$

When we choose $\beta = 1/2$, the non-Gaussianity parameter becomes

$$f_{\mathrm{NL}}^{\mathrm{equil}} = \frac{30}{9N^2} - \frac{5}{12N} \,. \tag{4.82}$$

In Figure 4.10 we present the behaviour of $f_{\mathrm{NL}}^{\mathrm{equil}}$, given by (4.81). Also, we show the non-Gaussianity parameter for the Starobinsky model, given by (4.80), and for the β-model with $\beta = 1/2$, given by (4.82), as functions of the number of e-folds.

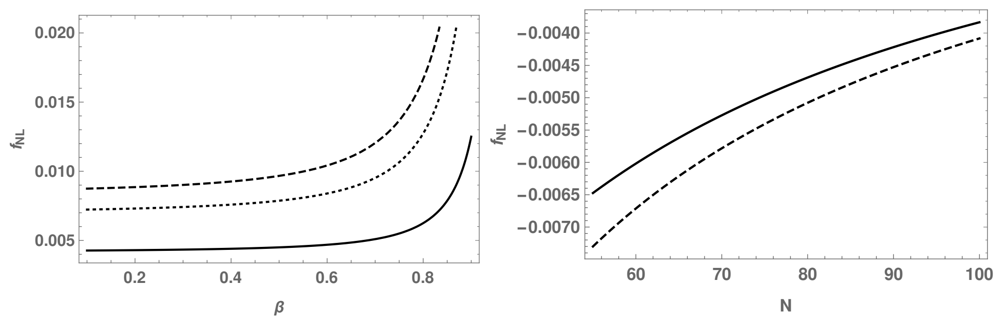


Fig. 4.10.: *Left Panel.* Non-gaussianity parameter for the β-model showing the cases : $N = 50$ (dashed line), $N = 60$ (dotted line) and $N = 100$ (solid line). *Right Panel.* Non-gaussianity parameter evolution for the Starobinsky model (dashed line) and the β-model (solid line), with $\beta = 1/2$, as functions of N.

It is important to highlight that our results are given for a single field model, which has a standard kinetic term, in the Einstein gravity and respects the slow roll conditions. So in order to obtain observably large non-gaussianities, it is necessary to violate one of these assumptions. There are several developments in this subject, since primordial non-gaussianities are one of the most promising probe of the physics of the early universe [90]. A recent interesting proposal is [91], where a generalisation of Maldacena's single-field result [88] to multifield models is obtained.

5 Alternative to Inflation

As discussed in the previous chapters, we know there is strong evidence for the existence of primordial density perturbations on large scales from cosmological observations. The standard explanation for the origin of the primordial perturbations is inflation, in which the universe underwent an accelerated expansion in the early universe. Although it is a well tested paradigm since its predictions on the primordial power spectrum can be successfully compared with observations of the CMB, inflation has some conceptual problems which motivate the exploration of alternative scenarios for the early universe. Among these challenges, we highlight

- The singularity problem, since even if inflation can be achieved by a scalar field coupled to Einstein gravity, this inflationary universe is past incomplete [92, 93]. This means that inflation cannot provide the whole history of the very early universe;

- The difficulty of constructing sufficiently flat potentials for inflation (or quasi de Sitter) solutions in string theory or supergravity [61, 62]. Even knowing that these swampland criteria have been severely criticised [94], it is worth to investigate whether or not inflation is the only consistent model for the origin of large-scale structure in our universe;

- The trans-Planckian problem for fluctuations, since if inflation lasts longer than the minimal amount of time necessary to solve the initial condition problems of Cosmology, then the wavelengths of cosmological scales originate at sub-Planckian values [95]. By knowing that in the trans-Planckian regime, General Relativity and Quantum Field Theory are not valid anymore, it is at least awkward considering the origin of cosmological perturbations in this regime;

Going further, we do not have a unique prescription for the initial conditions of our Universe, so there is no reason to avoid considering other mechanisms beyond inflation. Contracting Cosmologies, for example, can be a possible explanation in which the large scale structure of our Universe is generated from vacuum fluctuations before the Big Bang. Similarly to inflation, it is also possible to calculate a spectrum of perturbations on super-Hubble scales in the Hot Big Bang model assuming vacuum fluctuations on sub-Hubble scales in a preceding contracting phase. To make sure the spectrum calculated in the collapse phase is indeed related to perturbations in the standard Hot Big Bang, we need an intermediate bounce. The idea of a cosmological bounce is quite old (the first solutions were obtained in the late 70's [96, 97]), and it generically consists in considering the increasing of the Hubble rate, which emerges from the contracting phase with a negative value, until it becomes positive since it characterises the expanding phase. Hence, it is possible to have a causal generation mechanism for vacuum fluctuations. Obviously, the nature of a bounce depends on the specific model considered, for more information on general features and current status of Bouncing Cosmologies, see [98, 99, 100, 101, 102].

Unfortunately, it is beyond the scope of this thesis to describe a bounce mechanism. Our goal here is to provide some statements about the primordial perturbations inherited from a preceding collapse phase by including stochastic effects. So, in this chapter, which is based in [43], we first review the classical dynamics of a scalar field cosmology with an exponential potential, and we discuss the phase-space portrait for this theory as well as the stability of the fixed points which represent power law expansion or collapse. We study linear perturbations about the background solutions and in particular the solutions of the perturbed field in a collapsing scenario. Then, we describe the stochastic formalism, which modifies the classical dynamics of the scalar field above a coarse-graining scale, in order to apply it to study of the quantum fluctuations in the form of a stochastic noise on large scale. Finally, we study the deviations about the classical solution in phase-space, as well as the maximum lifetime of collapsing scenarios in the presence of stochastic fluctuations.

Collapsing universe

In order to set the initial conditions for inhomogeneous perturbations at early times, we will consider a canonical scalar field with energy density and pressure, given by (3.3), where

$V(\varphi)$ is the potential energy given by $V(\varphi) \propto e^{-\kappa\lambda\varphi}$, and λ is the slope of the potential. This provides the scaling solution[1]

$$P = w\rho \quad \text{where} \quad 3(1+w) = \lambda^2 \,. \tag{5.1}$$

It is possible to identify three scalar field collapse scenarios based on the form of the potential [103, 5, 27]

- Non-stiff collapse with $P < \rho \Leftrightarrow V > 0$ (including scale-invariant collapse);
- Pre-Big Bang collapse with $P = \rho \Leftrightarrow V = 0$ (blue tilted collapse);
- Ekpyrotic collapse with $P \gg \rho \Leftrightarrow V < 0$ (ultra-stiff fast roll collapse);

Background phase-space

In order to perform a qualitative analysis of the system described by (3.7) and (3.8), new variables can be introduced as [5]

$$x = \frac{\kappa\dot{\varphi}}{\sqrt{6}H}\,, \quad y = \frac{\kappa\sqrt{\pm V}}{\sqrt{3}H}, \tag{5.2}$$

where $H = \dot{a}/a$ is the Hubble rate and we use $\pm$ for positive and negative scalar potentials, $\pm V > 0$. With these variables, we can write the Friedmann constraint (3.8) as

$$x^2 \pm y^2 = 1\,, \tag{5.3}$$

Note that the equation of state (3.5) becomes

$$w = \frac{x^2 \mp y^2}{x^2 \pm y^2}\,. \tag{5.4}$$

[1]Scaling solutions are solutions in which the energy density of the scalar field mimics the background energy density. This means the energy density of the scalar field decreases proportionally to that of a barotropic perfect fluid.

Then, we are able to rewrite (3.7) in terms of the autonomous system

$$x' = -3x(1-x^2) \pm \lambda\sqrt{3/2}y^2 \ , \tag{5.5}$$

$$y' = xy(3x - \lambda\sqrt{3/2}) \ , \tag{5.6}$$

where a prime is a derivative with respect to the logarithm of the scalar factor, $N = \ln a$. We identify critical points of the system with fixed points where $x' = 0$ and $y' = 0$.

There are two kinetic-dominated solutions

$$x_a = -1 \text{ or } +1 \ , \quad y_a = 0 \ , \tag{5.7}$$

with equation of state Eq.(5.4) $w_a = 1$. These fixed points therefore correspond to solutions where $a \propto t^{1/3}$ in an expanding universe for $t > 0$, or $a \propto (-t)^{1/3}$ for $t < 0$ in a contracting universe.

There is also a potential-kinetic-scaling solution for $\pm(6 - \lambda^2) > 0$ (i.e., a sufficiently flat positive potential, $\lambda^2 < 6$ for $V > 0$, or a sufficiently steep negative potential, $\lambda^2 > 6$ for $V < 0$) which is given by

$$x_b = \frac{\lambda}{\sqrt{6}} \ , \quad y_b = \sqrt{\frac{\pm(6-\lambda^2)}{6}} \ . \tag{5.8}$$

This scaling solution, corresponds to a solution with constant equation of state (5.4)

$$w_b = \frac{\lambda^2}{3} - 1 \ , \tag{5.9}$$

and thus a power-law solution for the scale factor:

$$a(t) \propto |t|^p \ , \quad \varphi(t) = \sqrt{\frac{4}{3\kappa^2(1+w_b)}} \ln|t| + C \ , \tag{5.10}$$

where C is an arbitrary constant of integration and:

$$p = \frac{2}{\lambda^2} \, . \tag{5.11}$$

First-order perturbations around this critical point yield the linearised equation [5]

$$x' = \frac{(\lambda^2 - 6)}{2}(x - x_b) \; , \tag{5.12}$$

and its phase-space is described in Fig.5.1.

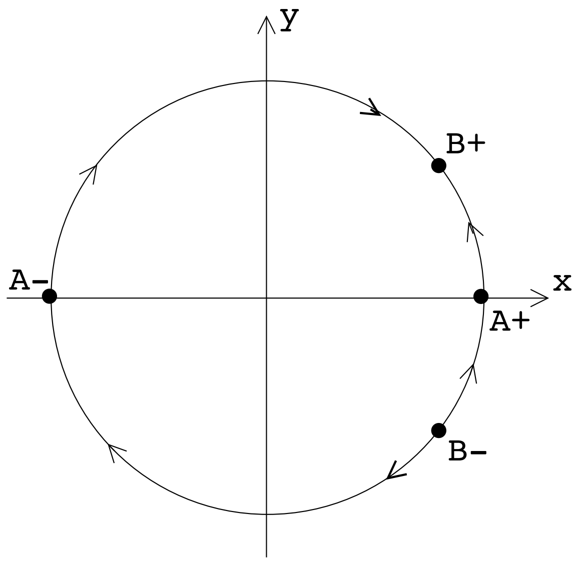


Fig. 5.1.: Phase-space for flat positive potentials, $\lambda^2 < 6$. Arrows indicate evolution in cosmic time, t. Image Credit: Ref.[5].

We see that in an expanding universe, $H > 0$, the scaling solution (5.8) is stable for $\lambda^2 < 6$, corresponding to $p > 1/3$ from (5.11). Thus the scaling solution is stable whenever it exists for a positive potential in an expanding universe, but it is never stable for a negative potential in an expanding universe. Conversely, in a collapsing universe, since N decreases with cosmic time, $H < 0$, the scaling solution is stable for $\lambda^2 > 6$, corresponding to $p < 1/3$. Thus the scaling solution is stable whenever it exists for a negative potential in a collapsing universe, but it is never stable for a positive potential for $H < 0$.

In summary:

- Expanding universe ($N \to +\infty$):
  - The scaling solution exists and is stable for a positive, flat potential $p > 1/3$ (including inflation, $p > 1$).
  - The scaling solution exists but is unstable for a negative, steep potential $p < 1/3$.
- Contracting universe ($N \to -\infty$):
  - The scaling solution exists and is stable for a negative steep potential $p < 1/3$ (including ekpyrosis, $p \ll 1$).
  - The scaling solution exists but is unstable for a positive flat potential $p > 1/3$ (including matter collapse, $p \simeq 2/3$).

Scalar field and metric perturbations

The configuration we chose, *i.e.*, a scalar field with exponential potential, allows us to explore the similarities between expanding and contracting scenarios. Then, we can apply the same method we used in Sec.3 to obtain the perturbed spectrum in collapsing universes, as show in [27].

Hence the solutions for small (sub-Hubble) and large (super-Hubble) scales are respectively

$$\delta\varphi \simeq \frac{e^{-ikt/a}}{a\sqrt{2k}} \quad \text{for } k^2/a^2 \gg H^2 \,, \tag{5.13}$$

$$\delta\varphi \simeq \frac{C\dot{\varphi}}{H} + \frac{D\dot{\varphi}}{H}\int \frac{H^2}{a^3\dot{\varphi}^2}dt \quad \text{for } k^2/a^2 \ll H^2 \,. \tag{5.14}$$

where we have chosen the quantum vacuum normalisation for the under-damped oscillations on sub-Hubble scales (5.13).

A characteristic feature of an inflating spacetime is that the comoving Hubble length decreases in an accelerating expansion with $\dot{a} > 0$ and $\ddot{a} > 0$. The same is true for the comoving Hubble length, $|H|^{-1}/a = 1/|\dot{a}|$ in a decelerating, collapsing universe with $\dot{a} < 0$ and $\ddot{a} < 0$.

As a result quantum vacuum fluctuations on sub-Hubble scales at early times (5.13), lead to well-defined predictions for the power spectrum of perturbations on super-Hubble scales for potential-kinetic-scaling solutions with $\lambda^2 < 2$ and hence $p > 1$ in an expanding cosmology, or with $\lambda^2 > 2$ and hence $p < 1$ in a collapsing cosmology.

The characteristics of the inflation and collapse models for different values of p are summarised in table 5.1.

Power-law inflation	Decelerated collapse
$H > 0$	$H < 0$
$\dot{a} > 0$, $\ddot{a} > 0$	$\dot{a} < 0$, $\ddot{a} < 0$
$p > 1$	$0 < p < 1$

Tab. 5.1.: Comparing the quantities H, $\dot{a}$, $\ddot{a}$ and p for power-law inflation and collapse. Although $\dot{a}$ is negative in the collapse case, its magnitude $|\dot{a}|$ is increasing. $p < 0$ is not allowed since this would requires $\rho_\varphi + P_\varphi < 0$.

Scalar field perturbations in a power-law collapse

Considering a collapsing universe with the scale factor being a power-law scaling solution with $a \propto (-t)^p$ and $t < 0$, we can reexpress the scale factor in terms of conformal time as

$$a(\eta) \propto (-\eta)^{p/(1-p)} , \tag{5.15}$$

Using the relation for the Hubble rate in conformal time, $H = a'/a^2$, we find that a can also be expressed as

$$a = \left(\frac{p}{1-p}\right)\frac{1}{H\eta}\,, \tag{5.16}$$

where $\eta < 0$.

Since $\dot{\varphi}/H$ is constant in this case we have $z \propto a$, which allows us to rewrite (3.26) as a Bessel equation

$$\frac{d^2v}{d\eta^2} + \left(k^2 - \frac{\nu^2 - 1/4}{\eta^2}\right)v = 0\,, \tag{5.17}$$

where

$$\nu = \frac{3}{2} + \frac{1}{p-1} = -\frac{3}{2}\left[\frac{1+(3p-2)}{1-(3p-2)}\right]\,. \tag{5.18}$$

Note that for power-law collapse with $p < 1$ we have $\nu < 3/2$[2].

Similarly as the procedure to derivate (3.32), we already know that the general solution for a given k can be expressed as a linear combination of Hankel functions, then we can normalise the solution with the Bunch-Davies vacuum on small scales, and find corresponding solution on large scales (late times) for $k\eta \to 0$

$$\delta\varphi_k = \frac{i}{a}\sqrt{\frac{1}{4\pi k}}\frac{\Gamma(|\nu|)2^{|\nu|}}{|k\eta|^{|\nu|-1/2}}\,. \tag{5.19}$$

As discussed in [27], this solution generates a scale invariant spectrum, $|\delta\varphi_k^2| \propto k^{-3}$, not only for slow-roll inflation ($P/\rho \to -1$ and $\nu = 3/2$) but also for a pressureless collapse ($P/\rho \to 0$ and $\nu = -3/2$).

[2]In Appendix B we explicitly show the mapping between the quantities p, ν, λ^2 and w.

Combining Eq.(5.19) and Eq.(5.15), we see that in this large-scale limit the field perturbations are constant for $\nu > 0$, since:

$$\delta\varphi_k \propto \frac{1}{a}(-\eta)^{\frac{1}{2}-|\nu|} \propto (-\eta)^{\nu-|\nu|} \,. \tag{5.20}$$

Conversely, for $\nu < 0$ we see that the scalar field perturbations can grow rapidly on super-Hubble scales and diverge as $\eta \to 0$.

Stochastic formalism

As we discussed in Chapter 3, we considered the homogeneous part of the scalar field to evolve classically, and its perturbations are treated as quantum mechanical quantities. An issue that could be raised in this scenario is: Can quantum corrections to the classical trajectory modify the background evolution? A possible answer for it can be found by using the stochastic approach. It models the effect of quantum vacuum fluctuations by introducing a cut-off scale (the so-called coarse-graining scale) splitting the fluctuations into two parts: quantum vacuum modes (below the coarse-graining scale) and the long wavelength field which includes a stochastic noise.

Since we would like to investigate how the equation of state of a inflationary or collapse cosmology can be modified by stochastic effects, we first dedicate this section to describe the stochastic formalism, and after apply it for the case of a scalar field with an exponential potential.

Stochastic noise from quantum fluctuations

We will keep working with a single scalar field φ with potential $V(\varphi)$, described by (3.1), for simplicity, but this analysis could be extended to a multiple-field configuration. Also, our analysis is made to scalar fluctuations only, so we are using the metric given by (3.18).

The Hamilton equations from the action (3.1) are given by

$$\frac{\partial \varphi}{\partial N} = \pi \ , \quad \frac{\partial \pi}{\partial N} = (\epsilon_H - 3)\pi - \frac{V_\varphi(\varphi)}{H^2} \ , \tag{5.21}$$

where the homogeneous background field φ and its conjugate momentum π are two independent dynamical variables needed to formulate the stochastic inflation (or collapse) in the full phase-space.

The crucial point of the stochastic formalism is to split the scalar field and its momentum into a long wavelength part and small-wavelength part as

$$\varphi = \overline{\varphi} + \varphi_Q \ , \qquad \pi = \overline{\pi} + \pi_Q \ , \tag{5.22}$$

where $\overline{\varphi}$ and $\overline{\pi}$ are the coarse-grained quantities, and the subscript "Q" describes the small-wavelength quantities

$$\varphi_Q = \int \frac{d^3k}{(2\pi)^{3/2}} W\left(\frac{k}{k_\sigma}\right) \left[a_{\vec{k}} \varphi_{\vec{k}}(N) e^{-i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} \varphi^\star_{\vec{k}}(N) e^{i\vec{k}\cdot\vec{x}}\right] \ , \tag{5.23}$$

$$\pi_Q = \int \frac{d^3k}{(2\pi)^{3/2}} W\left(\frac{k}{k_\sigma}\right) \left[a_{\vec{k}} \pi_{\vec{k}}(N) e^{-i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} \pi^\star_{\vec{k}}(N) e^{i\vec{k}\cdot\vec{x}}\right] \ , \tag{5.24}$$

here $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ are creation and annihilation operators[3], and W is a window function. This means that when its argument is small ($k/k_\sigma \ll 1$), $W \simeq 0$, and when its argument is large ($k/k_\sigma \gg 1$), $W \simeq 1$. The inclusion of this time-dependent comoving cut-off scale (the so-called coarse-graining scale)[4]

$$k_\sigma = \sigma a H \ , \tag{5.25}$$

allow us to derive an effective equation of motion for $\overline{\varphi}$, integrating out the degrees of freedom contained in φ_Q.

Applying the decompositions given by (5.22) into the classical equations for motion (5.21), the equations for the long-wavelength parts become stochastic Langevin equations for

[3] $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ satisfy the usual commutation relations $[a_{\vec{k}}, a^\dagger_{\vec{k}}] = \delta^3(\vec{k} - \vec{k}')$ and $[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}}] = [a_{\vec{k}}, a_{\vec{k}}] = 0$.
[4] $\sigma \ll 1$ is a fixed parameter that sets the coarse-graining scale [40].

the random field variables $\overline{\varphi}$ and $\overline{\pi}$

$$\frac{\partial \overline{\varphi}}{\partial N} = \overline{\pi} + \xi_\varphi \ , \tag{5.26}$$

$$\frac{\partial \overline{\pi}}{\partial N} = (\epsilon_H - 3)\overline{\pi} - \frac{V_\varphi(\overline{\varphi})}{H^2} + \xi_\pi \ , \tag{5.27}$$

where the quantum noises ξ_φ and ξ_π are given by

$$\xi_\varphi = -\int \frac{d^3k}{(2\pi)^{3/2}} \frac{dW}{dN}\left(\frac{k}{k_\sigma}\right)\left[a_{\vec{k}}\varphi_{\vec{k}}(N)e^{-i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\varphi^\star_{\vec{k}}(N)e^{i\vec{k}\cdot\vec{x}}\right] \ , \tag{5.28}$$

$$\xi_\pi = -\int \frac{d^3k}{(2\pi)^{3/2}} \frac{dW}{dN}\left(\frac{k}{k_\sigma}\right)\left[a_{\vec{k}}\pi_{\vec{k}}(N)e^{-i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\pi^\star_{\vec{k}}(N)e^{i\vec{k}\cdot\vec{x}}\right] \ . \tag{5.29}$$

This leads to stochastic noise associated with the small wavelength modes crossing the coarse-graining scale into the long-wavelength field at each time step, dN, described by the two-point correlation matrix $\Xi(\vec{x}_1, N_1; \vec{x}_2, N_2)$

$$\Xi_{f,g}(\vec{x}_1, N_1; \vec{x}_2, N_2) = \langle 0|\xi_f(\vec{x}_1, N_1)\xi_g(\vec{x}_2, N_2)|0\rangle \ , \tag{5.30}$$

with f, g and ξ_f, ξ_g being shorthand notation for the field or its momentum and their respective noises. Then, we can rewrite these entries in terms of the power spectrum for white noise [5] as [40]

$$\Xi_{f,g}(N) = \frac{1}{6\pi^2}\frac{dk_\sigma^3(N)}{dN} f_k(N)\, g^*_k(N) \ . \tag{5.31}$$

The stochastic approach provides a powerful way of calculation for quantum field effects on inflationary/collapsing spacetimes. Knowing that the presence of a stochastic noise can affect the stability of a dynamical system in a non-trivial way, our target now is to investigate how these stochastic perturbations can modify the equation of state of scalar field with an exponential potential in inflationary or collapsing scenario.

[5] such that $\langle\zeta(N)\rangle = 0$ and $\langle\zeta(N_1)\zeta(N_2)\rangle = \delta(N_1 - N_2)$.

Perturbations in phase-space variables

Introducing first order perturbations of the dimensionless phase-space variables (5.2), we obtain

$$\delta x = \frac{\kappa}{\sqrt{6}}\frac{1}{H}\left(\dot{\delta\varphi} - A\dot{\varphi} - \frac{\dot{\varphi}}{H}\delta H\right) , \tag{5.32}$$

$$\delta y = \frac{\kappa}{\sqrt{3}}\frac{\sqrt{V}}{H}\left(\frac{V_{,\varphi}}{2V}\delta\varphi - \frac{\delta H}{H}\right) , \tag{5.33}$$

where we are also including the metric perturbations $t \to (1+A)t$ and $H \to H + \delta H$ as described in [37]. By perturbing the Friedmann equation, we easily obtain

$$\delta H = \frac{\kappa^2}{6H}(V_{,\varphi}\delta\varphi + \dot{\varphi}\dot{\delta\varphi} - \dot{\varphi}^2 A) , \tag{5.34}$$

and since we are working in the spatially-flat gauge we can use the momentum constraint to write the perturbed lapse function in terms of the scalar field perturbation as

$$A = \frac{\kappa^2\dot{\varphi}\delta\varphi}{2H} . \tag{5.35}$$

At the critical point $x = x_b$ given by (5.8), the large-scale solution for the scalar field perturbations (5.19) then gives[6]

$$\delta x = \frac{i\kappa}{\sqrt{24\pi}}\left(1 - \frac{\lambda^2}{6}\right)\left(\frac{2}{2\nu - 1}\right)^2 \frac{\Gamma(|\nu|)2^{|\nu|}}{k^{|\nu|}}\left(|\nu| - \nu\right)\, H\left(-\eta\right)^{-|\nu|+3/2} , \tag{5.36}$$

To find the stochastic version of the first-order perturbations equation given by (5.12), given in terms of number of e-folds. Let us then re-express δx by setting a pivot scale

$$H = H_\star \exp\left[\left(\frac{\nu - 3/2}{\nu - 1/2}\right)(N_\star - N)\right] . \tag{5.37}$$

[6] The kinetic dominated solution analysis is briefly discussed in Appendix C, since it provides $\delta x = 0$ without regard to the solution $\delta\varphi$.

where $N = \ln a$ is the logarithmic expansion (or "e-folds") and the conformal time can be expressed using (5.15) as

$$(-\eta) = (-\eta_\star) \exp\left[\frac{1}{\nu - 1/2}(N_\star - N)\right] . \tag{5.38}$$

which gives:

$$\delta x = \frac{i\kappa}{\sqrt{24\pi}}\left(1 - \frac{\lambda^2}{6}\right)\left(\frac{2}{2\nu - 1}\right)^{2+|\nu|}\frac{\Gamma(|\nu|)2^{|\nu|}}{\sigma^{|\nu|}}\left(|\nu| - \nu\right) H_\star(-\eta)_\star^{3/2} \exp\left[\frac{\nu}{\nu - 1/2}(N_\star - N)\right] . \tag{5.39}$$

Perturbing the Friedmann constraint (5.3) leads to the expression for δy on large scales

$$\delta y = -\frac{x}{y}\delta x . \tag{5.40}$$

As a consequence, we can write the perturbation of the equation of state (5.4) as:

$$\delta w = 4x\delta x . \tag{5.41}$$

For adiabatic perturbations on large scales, which leave the equation of state unperturbed $\delta w = 0$, we have $\delta x = \delta y = 0$, i.e., adiabatic perturbations leave the critical points in the phase-space unperturbed. Adiabatic perturbations on large scales correspond to local perturbations forwards or backwards in time along the background trajectory [104].

Quantum diffusion in phase-space

Near the critical point $x = x_b$ given by (5.8), the stochastic version of (5.12) is[7]

$$\frac{d(\bar{x} - x_b)}{dN} = m(\bar{x} - x_b) + \hat{\xi}_x , \tag{5.42}$$

[7] For more informations on stochastic differential equations, see e.g. [105].

where the eigenvalue $m = (\lambda^2 - 6)/2$, whose solution is given by considering an Itô process

$$\bar{x}(N) - x_c = e^{m(N-N_\star)}\left(\bar{x}(N_\star) - x_c\right) + \int_{N_\star}^{N} e^{m(N-S)}\hat{\xi}_x dS. \tag{5.43}$$

We define the variance associated with the coarse-grained field $\bar{x}$ as

$$\sigma_x^2 := \left\langle \left(\bar{x}(N) - x_c\right)^2 \right\rangle \; ; \tag{5.44}$$

whose evolution equation is given by

$$\frac{d\sigma_x^2}{dN} = 2m\sigma_x^2 + 2\left\langle \hat{\xi}_x\left(\bar{x} - x_c\right)\right\rangle . \tag{5.45}$$

The solution can be split into a classical part and a quantum part, given by[8]

$$\begin{aligned}\sigma_x^2(N) &= \sigma_{x,cl}^2(N) + \sigma_{x,qu}^2(N)\\ &= \sigma_x^2(N_\star)e^{2m(N-N_\star)} + \int_{N_\star}^{N} dS\, e^{2m(N-S)}\Xi_{x,x}(S) \,,\end{aligned} \tag{5.46}$$

where the two-point correlation matrix $\Xi_{x,x}(S)$ is defined in Eq. (5.31), using the notation of [40]. The classical part, $\sigma_{x,cl}^2(N)$, is given by the variance at some initial time times an exponential function of the number of e-folds. The quantum part, $\sigma_{x,qu}^2(N)$, is the accumulated noise between the initial time and a later time.

We find the two-point correlation function for the perturbations in the dimensionless phase-space variable, x, by applying (5.31) to δx given in (5.39) as

$$\begin{aligned}\Xi_{x,x}(N) &= \frac{1}{2\pi^2}\sigma^3\left(\nu - \frac{1}{2}\right)^2 \frac{1}{(-\eta_\star)^3}\exp\left[\frac{-3}{\nu - 1/2}\left(N_\star - N\right)\right]|\delta x|^2\\ &= g(\nu,\sigma)\kappa^2 H^2(N) \,,\end{aligned} \tag{5.47}$$

with

$$g(\nu,\sigma) := \frac{\Gamma^2(|\nu|)\nu^2 2^{2|\nu|+4}}{(12\pi)^3\sigma^{2|\nu|-3}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|+4}\left(|\nu| - \nu\right)^2 . \tag{5.48}$$

[8]We show the explicit calculation in Appendix F.

As previously noted, for $\nu > 0$ the classical trajectory for x remains preserved by the leading order perturbations in the scalar field since $\delta x = 0$ on large scales $(k\eta \to 0)$ [9], but the same is not true for ν negative since in this case $\delta x \neq 0$. We derive in Appendix E an alternative way to find this result using perturbations of the field and momentum.

Inserting our result for the correlation function (5.47) in the quantum part of (5.46), we find

$$\sigma^2_{x,qu}(N) = g(\nu,\sigma)\kappa^2 H_\star^2 \exp\left[\frac{3-2\nu}{\nu-1/2}(-N_\star)\right] e^{2mN}\int_{N_\star}^{N} dS\, e^{-2mS}\exp\left[\frac{3-2\nu}{\nu-1/2}S\right] . \tag{5.49}$$

Re-expressing the eigenvalue m in terms of the index, $m = -2\nu/(\nu - 1/2)$, the solution of (5.49) is then

$$\sigma^2_{x,qu}(N) = \tilde{g}(\nu,\sigma)\kappa^2 H^2(N)\left\{1 - \exp\left[\frac{3+2\nu}{\nu-1/2}(N_\star - N)\right]\right\} , \tag{5.50}$$

where

$$\begin{aligned} \tilde{g}(\nu,\sigma) &= \left(\frac{\nu-1/2}{3+2\nu}\right) g(\nu,\sigma) \\ &= \frac{\Gamma^2(|\nu|)2^{2|\nu|+4}}{(12\pi)^3\sigma^{2|\nu|-3}}\left(\frac{\nu^2}{3+2\nu}\right)\left(\frac{2}{2\nu-1}\right)^{2|\nu|+3}(|\nu|-\nu)^2 . \end{aligned} \tag{5.51}$$

Equation (5.49) is given in terms of the Hubble scale at a fixed time, $H_\star$, while we have used (5.37) to give the variance (5.50) in terms of the time-dependent Hubble scale, $H(N)$.

We can compare the growth rate of the classical and quantum perturbations by comparing the time dependence from the two parts in (5.46). We note first that the time dependence of the classical term goes as

$$\sigma_{x,cl} \propto \exp\left[\frac{4\nu}{\nu-1/2}(N_\star - N)\right] . \tag{5.52}$$

[9] In Appendix D, we take into account the next-to-leading order field contribution to compute $\Xi_{x,x}(N)$ and show that even in this configuration quantum diffusion should not take the system away from the fixed point. We show in particular that this is the case for quasi-de Sitter inflation.

From (5.50), we see the time dependence of the quantum term behaves as

$$\sigma_{x,qu} \propto \exp\left[\frac{4\nu}{\nu-1/2}(N_\star - N)\right]\left\{\exp\left[-\frac{3+2\nu}{\nu-1/2}(N_\star - N)\right]-1\right\} . \tag{5.53}$$

Remember $N_\star - N$ grows with time (N decreases) in an expanding universe. Thus, the quantum variance decays with time if we have

$$\frac{3+2\nu}{\nu-1/2} > 0 . \tag{5.54}$$

This is the case if either $\nu > 1/2$ or $\nu < -3/2$. However, a positive ν will cancel the leading order quantum diffusion, so we will consider only the second case, $\nu < -3/2$, in the following analysis. Thus, the classical perturbations grow faster than the quantum noise if $\nu < -3/2$, and the quantum noise grows faster if $\nu > -3/2$.

Also, the condition (5.54) provides a shift in the spectrum, when compared to the case $\nu = -3/2$, since the scalar spectral index can be written in terms as [106, 107, 108, 109, 99, 110]

$$n_s = 1 + \frac{12w}{1+3w} = 1 + \frac{4(2\nu+3)}{3} , \tag{5.55}$$

and it is clear to see that when $\nu = -3/2 - \epsilon$, where ϵ is a small positive parameter, $w < 0$ and the spectrum becomes red, *i.e.*, $n_s < 1$. In Fig.(5.2), we show what values for ν are in agreement with Planck data.

To understand the behaviour around $\nu \approx -3/2$ we will consider $\nu = -3/2 - \epsilon$ which for $|\epsilon(N_\star - N)| \ll 1$ leads to

$$\sigma^2_{x,qu}(N) = \frac{3}{128\pi}\frac{1}{\sigma^{2\epsilon}}\frac{H^2(N)}{M^2_{pl}}(N_\star - N) , \tag{5.56}$$

where we have used $\kappa^2 = 8\pi/M^2_{pl}$ with M_{pl} the Planck mass, and we recall that σ is the coarse-graining scale. The diffusion thus has the form of a random walk with $N_\star - N$ steps of equal, but growing, length $\propto |H(N)|$.

We see that (5.56) depends weakly on the coarse-graining scale for $\nu \approx -3/2$, and

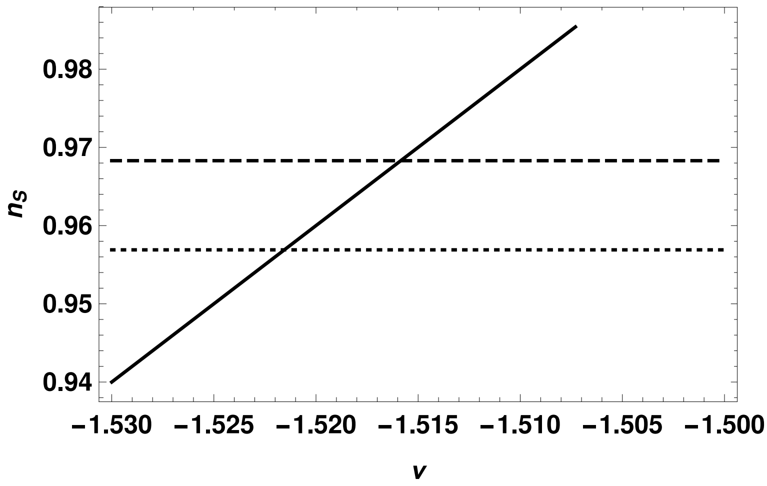


Fig. 5.2.: Evolution for n_s as function of ν. The horizontal dotted lines enclose the 68% confidence level of the values of n_s measured by Planck collaboration 2018.

becomes independent of σ in the limit $\nu = -3/2$, where $\epsilon \to 0$. This is not surprising since we know that quantum fluctuations in a pressureless collapse give rise to a scale-invariant spectrum of perturbations [107, 108, 111].

Maximum lifetime of the collapsing phase

We can now examine when the variance becomes large, i.e., when $\sigma_{x,qu} \approx 1$, so that the quantum diffusion due to the stochastic noise results in a significant deviation from the critical point.

- **Radiation-dominated collapse**: Consider first the case of a potential-kinetic-scaling collapse with $\lambda = 2$, giving rise to an equation of state $w = 1/3$, analogous to a radiation-dominated cosmology, and index $\nu = -1/2$. The variance (5.50) in this case becomes

$$\begin{aligned}\sigma^2_{x,qu}(N) &= \frac{\sigma^2}{54\pi}\frac{H_\star^2}{M_{pl}^2}\left\{\exp\left[4\left(N_\star - N\right)\right] - \exp\left[2\left(N_\star - N\right)\right]\right\} \\ &\approx \frac{\sigma^2}{54\pi}\frac{H^2(N)}{M_{pl}^2}\,, \end{aligned} \tag{5.57}$$

where to get the second line, we have neglected the second exponential term since the first one will grow much quicker. For $\sigma^2_{x,qu}(N_{\rm end}) = 1$, we get the straightforward result

$$|H_{\rm end}| \approx \frac{13}{\sigma} M_{pl} \ . \tag{5.58}$$

We conclude that a radiation-dominated collapsing phase cannot escape the fixed point due to quantum diffusion until it approaches the Planck scale. Indeed, since we require $\sigma < 1$, we see that a deviation from the classical fixed point $x = x_b$ due to quantum diffusion would require the Hubble scale to become greater than the Planck scale. In practice as soon as the Hubble scale approaches the Planck scale our semi-classical analysis breaks down;

- **Pressureless collapse**: For the case of a pressureless collapse, $\nu = -3/2$, we find $\sigma_{x,qu}(N_{\rm end}) = 1$ when

$$|H_{\rm end}| = \sqrt{\frac{128\pi}{3(N_\star - N_{\rm end})}} M_{pl} \tag{5.59}$$

Thus, for pressureless collapse case, quantum diffusion gives a time, $t_{\rm end}$, at which stochastic trajectories leave the classical fixed point before we reach the Planck scale, $|H_{\rm end}| < M_{pl}$, if the number of e-folds during the collapse is greater than 134. We show a simple example in Fig.(5.3). In terms of the initial Hubble rate, using (5.37) for the Hubble rate $H(N)$ for $\nu = -3/2$, we have

$$(N_\star - N_{\rm end}) \exp\left[3(N_\star - N_{\rm end})\right] = \frac{128\pi}{3} \frac{M_{pl}^2}{H_\star^2} \ , \tag{5.60}$$

from which we get an approximate number of e-folds during the collapse phase

$$N_\star - N_{\rm end} \approx \frac{2}{3} \ln\left(\sqrt{\frac{128\pi}{3}} \frac{M_{pl}}{|H_\star|}\right) \ . \tag{5.61}$$

Conversely we can obtain an expression for the Hubble rate at the end of the pressureless collapse starting from an initial Hubble rate $H_\star$ given by

$$|H_{\rm end}| \approx \sqrt{\frac{64\pi}{\ln\left(\sqrt{\frac{128\pi}{3}} \frac{M_{pl}}{|H_\star|}\right)}} M_{pl} \ . \tag{5.62}$$

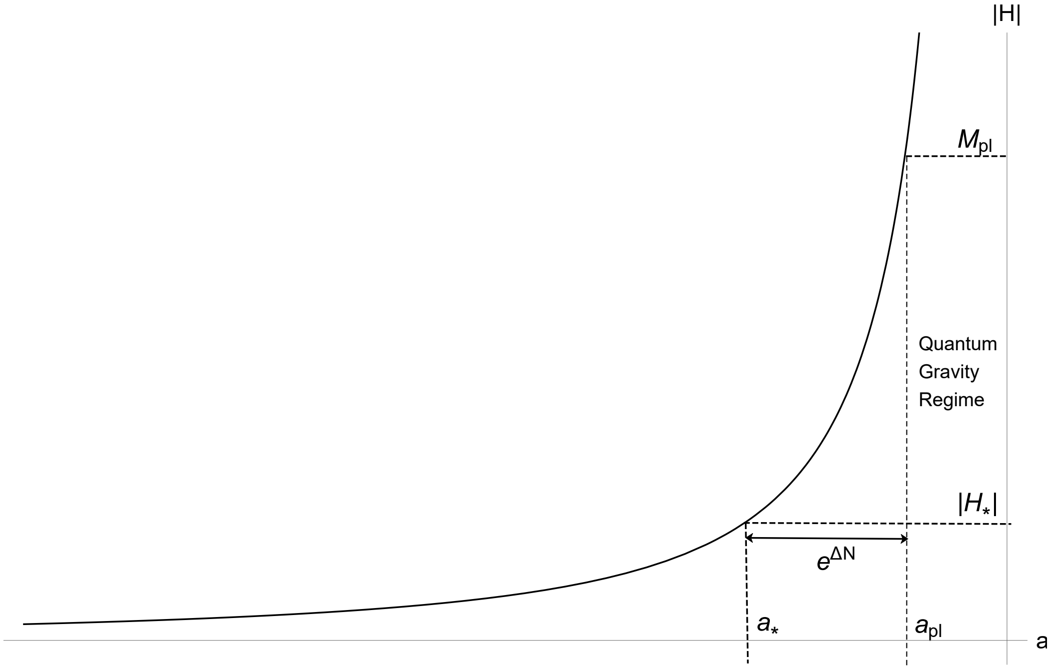


Fig. 5.3.: Evolution of the Hubble rate, $|H|$, in a pressureless collapse. For quantum diffusion to lead to a deviation from the classical fixed point before the Hubble rate reaches the Planck scale, $|H_{\rm end}| < M_{\rm Pl}$, requires a very low initial energy scale, $|H_*| \ll |H_{\rm end}|$.

Knowing how the comoving Hubble length behaves in terms of time during pressureless collapse, we can estimate a lower limit on the number of e-folds required during pressureless collapse to solve the horizon and smoothness problems of the hot big bang:

$$\frac{k_{\rm end}}{k_\star} = \frac{a_{\rm end}H_{\rm end}}{a_\star H_\star} = \left(\frac{t_\star}{t_{\rm end}}\right)^{1/3} = e^{(N_\star - N_{\rm end})/2} > e^{70} \; , \tag{5.63}$$

where we are considering 70 as a ratio between the Hubble length over Planck scale compared to horizon size today. This is a similar number for inflation to solve the flatness and horizon problems of Big Bang cosmology. Then, we would need

$$N_\star - N_{\rm end} > 140 \; , \tag{5.64}$$

which is remarkably close to the estimate $N_\star - N_{\rm end} > 134$ that follows from requiring $|H_{\rm end}| < M_{pl}$ (5.59);

6 Final Considerations

The first objective of this thesis was to derive observational constraints on inflationary models that arise from $f(R)$ theories. Motivated by the good agreement between the cosmological parameters n_s and r provided by the α-Attractors theory and recent observations, in Section 4.1, we derive a differential equation for a $f(R)$ theory from the scalar potentials of α-Attractors, (4.11), in order to find a $f(R)$ theory compatible with this class of models. Since this differential equation cannot be solved analytically for any choice α, we make an asymptotic analysis at high energies, where inflation is expected to take place, and show that the *ansatz* (4.22) represents a viable solution to the differential equation (4.11) at the leading and sub-leading order in the limit $R \to \infty$. Then, we investigate the predictions on the inflationary parameters provided by the power law extension of the Starobinsky model and show that they are in very good agreement with observation, allowing for a greater production of gravitational waves, i.e. a larger r, than in the Starobinsky model. We also made a detailed analysis of the model $f(R) = \gamma R^{2-\delta}$ and found a mostly horizontal movement of the predictions of the model in the r–n_s plane, indicating that the missing of the R^2 term in the action alters drastically the behaviour of the inflationary evolution.

The reconstruction here performed of the α-Attractors models is confined to a $f(R)$ model. Another investigation in this sense could start from a more general extension of GR, such as the Horndeski theory [112].

In Section 4.2, we also present formulae for the inflationary slow roll parameters (ϵ, η and ζ) and the non-Gaussianity function $f_{\rm NL}^{\rm equil}$ to be used in any kind of $f(R)$ proposals which satisfies the condition for slow roll existence, *i.e.*, any $f(R)$ that behaves like $f(R)/R^2 \approx constant$. As an example, we develop these quantities using the Starobinsky model and the β-model, which is a $f(R)$ approximate reconstruction of the α-Attractors class of inflationary models. As shown in Figure 4.10, these two models provide small non-Gaussianity parameters, which is in good agreement with recent constraints [89], that despite the agreement of the Planck data sets with the Gaussian scenario, they do not rule out the presence of non-Gaussianities of small intensity.

Since the origin of the vacuum fluctuations leading to the large-scale structures we observe in the Universe is still unknown, the second goal of thesis was the study of such fluctuations in a collapse phase. In Chapter 5, we considered three possible cosmological scenarios: a power-law inflation, a runaway collapse and the ekpyrotic collapse. It is well known that inflation and the ekpyrotic models are classically stable under perturbations, while the matter collapse is unstable. We showed that the quantum perturbations do not drive inflationary and ekpyrotic models away from their fixed point in phase-space, what in turn means one does not modify the classical behaviour of such models by adding quantum noises. In fact, all models with a positive index ν (index of the Bessel function governing the evolution of the Mukhanov-Sasaki variable) have an identically vanishing quantum contribution. As a consequence, only models with a negative index, including the pressureless collapse, see their classical equation of state being pushed away from the critical point. The perturbation of the equation of state, or equivalently of the kinetic variable x we used throughout this paper, is summarised in Table (6.1).

Finally, we studied the maximum lifetime of the quantum noise for a negative ν in a collapse phase. The general case seems to show that collapse models are stable against quantum diffusion. However, for the interesting value $\nu = -3/2$ corresponding to a pressureless collapse, known to be scale-invariant, we found the quantum diffusion takes us away from the critical point if we start the collapse from very low energy scales and if it lasts more than 134 e-folds, with no dependence on the coarse-graining scale. Even though for a value of ν arbitrarily close to $-3/2$ we recover some dependence on the coarse-graining scale, this dependence is found to be weak.

Inflation / Ekpyrotic collapse	Pressureless collapse
$\nu > 0 \to$ adiabatic noise	$\nu < 0 \to$ non-adiabatic noise
$\delta w = 0$	$\delta w \neq 0$

Tab. 6.1.: Table comparing the behaviour of δw for three cases in the limit $\sigma \to 0$: de Sitter inflation ($\nu = 3/2$), ekpyrotic collapse ($\nu = 1/2$) and pressureless collapse ($\nu = -3/2$).

As we focused on stochastic effects in a collapsing universe, the question of the perturbations fate through the singularity naturally emerges in this context. However, it is not possible to model a cosmological bounce in our approach as we have used the number of e-folds as time variable, which is not well suited to tackle this issue. A way-out could be the introduction of stochastic perturbations in the scale factor, while the field itself becomes the time variable. Also, we have not taken into account the gauge corrections arising from working out the field perturbations in conformal time and their subsequent use in the stochastic noise in terms of number of e-folds, so it would be interesting to use the stochastic formalism beyond the usual slow-roll approximation [37] to study this gauge issue in collapsing scenarios.

Appendices

A

Detailed investigation of the differential equation (4.11)

In this appendix we investigate in greater detail (4.11), which we report here

$$Rf' - f = \frac{3M^2}{2(1-\beta)^2}\left(f' - f'^{\beta}\right)^2 . \tag{A.1}$$

First of all, we define $f(R) \equiv R + F(R)$, i.e. we postulate the presence of the Einstein-Hilbert term and focus our attention just on its correction. Equation (A.1) thus becomes

$$RF' - F = \frac{3M^2}{2(1-\beta)^2}(1+F')^2\left[1-(1+F')^{\beta-1}\right]^2 . \tag{A.2}$$

Next, we normalise R and F to $3M^2/2$ and define $\gamma \equiv \beta - 1 = -1/\sqrt{\alpha} < 0$, thus obtaining

$$RF' - F = \frac{(1+F')^2}{\gamma^2}\left[1-(1+F')^{\gamma}\right]^2 . \tag{A.3}$$

Deriving this equation with respect to R, we obtain

$$\frac{2\left(F'+1\right)\left[\left(F'+1\right)^{\gamma}-1\right]\left[\left(\gamma+1\right)\left(F'+1\right)^{\gamma}-1\right]}{\gamma^2}F'' = RF'' . \tag{A.4}$$

Now, this equation is satisfied if $F'' = 0$ but then it would give a linear solution for $F(R)$ in which we are not interested since we have already stipulated that $f(R) \equiv R + F(R)$. Therefore, we assume that $F'' > 0$, and the above equation gives us

$$R = \frac{2\left(F'+1\right)\left[\left(F'+1\right)^{\gamma}-1\right]\left[\left(\gamma+1\right)\left(F'+1\right)^{\gamma}-1\right]}{\gamma^2} . \tag{A.5}$$

For $\gamma = -1$, one can easily recover the Starobinsky case, since

$$R = 2F' \quad \Rightarrow \quad F = \frac{R^2}{4} , \tag{A.6}$$

and restoring the normalisation by $3M^2/2$ one obtains $R + R^2/6M^2$.

Equation (A.5) gives us a constraint on the expression of $F'(R)$ which we must take into account when solving (A.3) in order to select the correct initial condition and exclude the linear solution. Note that $F'=0$ implies from (A.5) that $R=0$ and then, from (A.3), that $F=0$. We have thus that $F(0)=F'(0)=0$, as desired.

The strategy in order to numerically obtain a solution $F(R)$ is the following. We choose an initial, small but non vanishing $R_i \equiv \epsilon$; determine $F'(\epsilon)$ from (A.5); determine the initial condition $F(\epsilon)$ from (A.3); solve either (A.3) or (A.5). A simpler way is to use the fact that $F'(R) \to 0$ for $R \to 0$. Then, for a sufficiently small ϵ, keeping the lowest order in $F'(\epsilon)$ in (A.5) and (A.3), we get

$$\epsilon \sim 2F'(\epsilon)\ , \qquad F(\epsilon) = \frac{\epsilon^2}{4}\ , \tag{A.7}$$

which is indeed the Starobinsky case found in (A.6), revealing thereby that the Starobinsky model is a low curvature approximation of the α-Attractors. This can be also seen by taking the low χ limit of (4.10). We can also prove that the Starobinsky model is a high curvature approximation of the α-Attractors, since $F'' > 0$ and $\gamma < 0$. Then, for $R \to \infty$ we get from (A.5)

$$R \sim \frac{2F'}{\gamma^2}\ , \quad \text{for } R \to \infty\ , \quad \Rightarrow \quad F \sim \frac{\gamma^2}{4}R^2\ , \quad \text{for } R \to \infty\ . \tag{A.8}$$

Being $\gamma = \beta - 1$, restoring the $3M^2/2$ normalisation, one easily recover our approximation (4.22). Indeed, the plots in the upper panel of Fig. A.1 suggest that the Starobinsky model is also a high curvature approximation of the α-Attractors (of course, with different energy scales depending on the parameter α). This fact is somehow expected, since both the theories are characterised by a plateau potential at high energies.

In Fig. A.2 we also display the goodness of our approximation (4.22) in the case $\alpha = 4$.

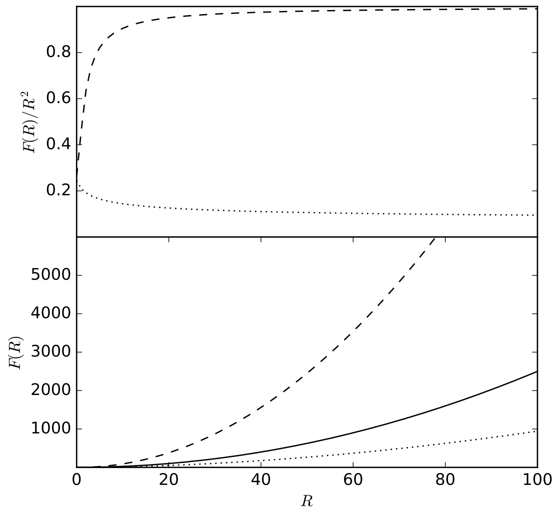


Fig. A.1.: *Upper Panel.* Evolution of the numerically reconstructed $F(R)$ normalised to R^2 for the cases $\gamma = -2$ ($\alpha = 1/4$, dashed line) and $\gamma = -1/2$ ($\alpha = 4$, dotted line). This plot shows an asymptotic quadratic behaviour.
Lower Panel. Reconstructed $F(R)$ for the cases $\gamma = -2$ ($\alpha = 1/4$, dashed line), the Starobinsky model $\gamma = -1$ ($\alpha = 1$, solid line) and $\gamma = -1/2$ ($\alpha = 4$, dotted line).

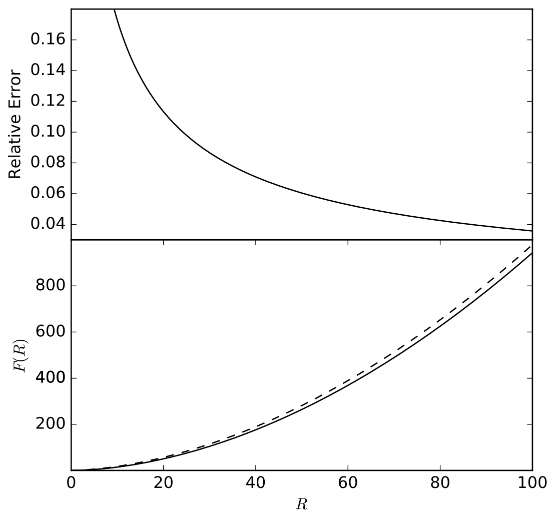


Fig. A.2.: *Upper Panel.* Relative error, showing the goodness of our approximation Eq. (4.22).
Lower Panel. Comparison between the numerical reconstructed $F(R)$ using the techniques of this section and the approximated one in Eq. (4.22), for $\gamma = -1/2$ (corresponding to $\beta = 1/2$ and $\alpha = 4$).

B Mapping between p, λ^2, w, ν and $n_s - 1$

Throughout this work, we use the quantities p, λ^2, w and ν because, even if they are connected to the others, each one of them is more appropriate for a specific analysis. In order to facilitate the understanding of the reader, we show the explicit mapping between them in Table B.1.

	p	λ^2	w	ν	$n_s - 1$
p	p	$\frac{2}{\lambda^2}$	$\frac{2}{3(1+w)}$	$\frac{2\nu-1}{2\nu-3}$	$\frac{4-(n_s-1)}{6-(n_s-1)}$
λ^2	$\frac{2}{p}$	λ^2	$3(1+w)$	$\frac{4\nu-6}{2\nu-1}$	$\frac{2(n_s-1)-12}{(n_s-1)-4}$
w	$\frac{2-3p}{3p}$	$\frac{\lambda^2-3}{3}$	w	$\frac{-2\nu-3}{6\nu-3}$	$\frac{(n_s-1)}{12-3(n_s-1)}$
ν	$\frac{3}{2}+\frac{1}{p-1}$	$\frac{3}{2}+\frac{\lambda^2}{2-\lambda^2}$	$\frac{3}{2}-\frac{3(1+w)}{1+3w}$	ν	$\frac{3(n_s-1)}{8}-\frac{3}{2}$
$n_s - 1$	$\frac{6p-4}{p-1}$	$\frac{4(3-\lambda^2)}{2-\lambda^2}$	$\frac{12w}{1+3w}$	$\frac{4(2\nu+3)}{3}$	$n_s - 1$

Tab. B.1.: Table showing how to write p, λ^2, w, ν and $n_s - 1$ in terms of each of them.

Kinetic-dominated solution

C

We show in this appendix the solutions for the other critical point, namely the kinetic-dominated regime. This is interesting for two reasons. First, this regime corresponds to the critical value $\nu = 0$, which is the interface between purely adiabatic perturbations (at first-order) and non-adiabatic perturbations. Second, perturbations in this regime act as a stiff fluid and go as $a \propto (-\eta)^{-6}$. Such behaviour is usually invoked in the classical resolution of the initial singularity, see for instance the reviews [100, 113, 101, 114].

To begin, note we can rewrite (5.32) in terms of the variables x and y as

$$\delta x = \frac{\kappa}{6}\left[\left(1 - x^2\right)\frac{\dot{\delta\varphi}}{H} + \left(3x^4 - 3x^2 + \frac{\lambda^2}{2}y^2\right)\delta\varphi\right] . \tag{C.1}$$

In the case of the kinetic-dominated solution, the fixed points are $x_a = \pm 1$, $y_a = 0$. In this configuration, we have $w = 1$, or equivalently $\lambda^2 = 6$, resulting in the trivial expression

$$\delta x = 0 \, , \tag{C.2}$$

regardless of the value of the solution $\delta\varphi$. Hence, any first-order linear field perturbation leads to adiabatic perturbations in the kinetic-dominated regime.

D Next-to-leading order field contribution

We expand the field solution (5.19) and its derivative to third order to get all terms contributing to second order. Then the field is now

$$\begin{aligned}
\delta\varphi &= \frac{i}{a}\frac{\Gamma\left(|\nu|\right)2^{|\nu|}}{\sqrt{4\pi}k^{|\nu|}}\left[1+\frac{(-k\eta)^2}{4\left(|\nu|-1\right)}+\frac{(-k\eta)^4}{32\left(|\nu|-1\right)\left(|\nu|-2\right)}\right]\frac{1}{(-\eta)^{|\nu|-1/2}} \\
&= \frac{i}{\sqrt{4\pi}}\left(\frac{2}{2\nu-1}\right)\frac{2^{|\nu|}\Gamma(|\nu|)}{k^{|\nu|}} \\
&\quad\times\left[\frac{H}{(-\eta)^{|\nu|-3/2}}+\frac{k^2H}{4\left(|\nu|-1\right)(-\eta)^{|\nu|-7/2}}+\frac{k^4H}{32\left(|\nu|-1\right)\left(|\nu|-2\right)(-\eta)^{|\nu|-11/2}}\right],
\end{aligned} \tag{D.1}$$

$$\begin{aligned}
\delta\dot{\varphi} &= \frac{i}{\sqrt{4\pi}}\left(\frac{2}{2\nu-1}\right)^2\frac{2^{|\nu|}\Gamma(|\nu|)}{k^{|\nu|}}H^2 \\
&\quad\times\left[\frac{\left(|\nu|-\nu\right)}{(-\eta)^{|\nu|-3/2}}+\frac{k^2\left(|\nu|-\nu-2\right)}{4\left(|\nu|-1\right)(-\eta)^{|\nu|-7/2}}++\frac{k^4\left(|\nu|-\nu-4\right)}{32\left(|\nu|-1\right)\left(|\nu|-2\right)(-\eta)^{|\nu|-11/2}}\right].
\end{aligned} \tag{D.2}$$

The contribution to the noise in x becomes

$$\Xi_{x,x}(N)=\bar{g}(\nu,\lambda,\sigma)\kappa^2H_\star^2\exp\left[-\frac{3-2\nu}{\nu-1/2}\left(N_\star-N\right)\right], \tag{D.3}$$

with

$$\begin{aligned}
\bar{g}(\nu,\lambda,\sigma):=&\frac{\Gamma^2(|\nu|)2^{2|\nu|}}{(12\pi)^3}\frac{(6-\lambda^2)^2}{\sigma^{2|\nu|-3}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|+2}\left[\left(|\nu|-\nu\right)^2+\frac{\sigma^2}{2}\frac{\left(|\nu|-\nu\right)\left(|\nu|-\nu-2\right)}{\left(|\nu|-1\right)}\left(\nu-\frac{1}{2}\right)^2\right. \\
&\left.+\frac{\sigma^4}{16}\frac{\left[\left(|\nu|-\nu\right)^2-4\left(|\nu|-\nu\right)\right]\left[2|\nu|-3\right]+4\left(|\nu|-2\right)}{\left(|\nu|-1\right)^2\left(|\nu|-2\right)}\left(\nu-\frac{1}{2}\right)^4\right]
\end{aligned} \tag{D.4}$$

For positive ν, the above equation is simplified to

$$\bar{g}(\nu>0,\lambda,\sigma):=\frac{\Gamma^2(|\nu|)2^{2|\nu|}}{(12\pi)^3}\frac{(6-\lambda^2)^2}{\sigma^{2|\nu|-3}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|+2}\left[\frac{\sigma^4}{4}\frac{1}{\left(|\nu|-1\right)^2}\left(\nu-\frac{1}{2}\right)^4\right] \tag{D.5}$$

The variance in the case of quasi-de Sitter inflation ($\nu = 3/2 - \epsilon$) is given by

$$\sigma^2_{x,qu} = \frac{1}{24\pi^2} H_\star^2 \kappa^2 \sigma^{4-\epsilon} (1 + 4\epsilon) \left[1 - \exp\left(-6(N - N_\star)\right)\right] \ . \tag{D.6}$$

Since N is growing with time the exponential vanishes quickly and eventually the time at which $\sigma^2_{x,qu} = 1$ is

$$H(N) = \sqrt{3\pi}\,(1 - 2\epsilon)\,\frac{M_{pl}}{\sigma^{2-\epsilon/2}} \ , \tag{D.7}$$

and since $\sigma < 1$ we see quantum diffusion drives us away only if the Hubble rate is far above the Planck scale. We note this result stays true for $N \approx N_\star$ since in this case we have

$$H(N) = \sqrt{\frac{\pi}{2}}\,(1 - 2\epsilon)\,\frac{M_{pl}}{\sigma^{2-\epsilon/2}} \ . \tag{D.8}$$

E

A different approach to find the noise

We know that $\dot{\varphi}$ can be related to its momentum $\pi \equiv \partial\mathcal{L}/\partial\dot{\varphi}$ using the ADM formalism, as shown in [40], by

$$\dot{\varphi} = \frac{1}{a(t)^3}\pi_\varphi \ , \tag{E.1}$$

where the lapse function N was chosen as the cosmic time, which means that $N = 1$. Also, the evolution for π_φ is given by

$$\dot{\pi}_\varphi = -a(t)^3 V_{,\varphi} \ , \tag{E.2}$$

where ",φ" represents the derivative with respect to φ.

We get the linearly perturbed momentum including scalar field perturbations

$$\pi + \delta\pi = \frac{\partial\left(\mathcal{L} + \delta\mathcal{L}\right)}{\partial\left(\frac{1}{1+A}\dot{\varphi}\right)} = (1+A)\,\frac{\partial\left(\mathcal{L} + \delta\mathcal{L}\right)}{\partial\dot{\varphi}} \tag{E.3}$$

where we have also perturbed the lapse function $t \to (1+A)t$. The perturbed momentum is then

$$\begin{aligned} \delta\pi &= A\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} + \frac{\partial\delta\mathcal{L}}{\partial\dot{\varphi}} \\ &= a^3\left(\dot{\delta\varphi} - A\dot{\varphi}\right) \end{aligned} \tag{E.4}$$

We may use the constraint $A = \kappa^2\dot{\varphi}\delta\varphi/2H$ to eliminate the perturbed lapse function since we are working in the spatially-flat gauge [37].

Using the definition of the coarse-graining scale (5.25) in the expressions for the field and its conjugate momentum

$$\delta\varphi = \frac{i}{\sqrt{4\pi}}\left(\frac{2}{2\nu-1}\right)\frac{2^{|\nu|}\Gamma(|\nu|)}{k^{|\nu|}}\frac{H}{(-\eta)^{|\nu|-3/2}}\,, \tag{E.5}$$

$$\delta\pi_\varphi = \frac{i}{\sqrt{4\pi}}\frac{2^{|\nu|}\Gamma(|\nu|)}{k^{|\nu|}}\left(\nu-\frac{1}{2}\right)^4\left[\left(\frac{2}{2\nu-1}\right)(|\nu|-\nu)-\frac{\kappa^2\dot{\varphi}^2}{2H^2}\right]\frac{1}{H(-\eta)^{|\nu|+3/2}}\,, \tag{E.6}$$

we get

$$|\delta\varphi|^2 = \frac{\Gamma^2(|\nu|)2^{2|\nu|}H^2}{4\pi\sigma^{2|\nu|}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|+2}\frac{1}{(-\eta)^{-3}}\,, \tag{E.7}$$

$$|\delta\pi|^2 = \frac{\Gamma^2(|\nu|)2^{2|\nu|}}{4\pi\sigma^{2|\nu|}H^2}\left(\frac{2}{2\nu-1}\right)^{2|\nu|-4}\left[\left(\frac{2}{2\nu-1}\right)(|\nu|-\nu)-4\pi G\frac{\dot{\varphi}^2}{H^2}\right]^2\frac{1}{(-\eta)^3}\,, \tag{E.8}$$

$$\delta\varphi\delta\pi^* = \frac{\Gamma^2(|\nu|)2^{2|\nu|}}{4\pi\sigma^{2|\nu|}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|-1}\left[\left(\frac{2}{2\nu-1}\right)(|\nu|-\nu)-4\pi G\frac{\dot{\varphi}^2}{H^2}\right]\,. \tag{E.9}$$

From this we can work out the two-points correlation matrix associated with the quantum noise with respect to conformal time ξ_φ and ξ_π [40] .

$$\Xi^{(\eta)}_{\varphi,\varphi} = \frac{1}{(2\pi)^3}\frac{\Gamma^2(|\nu|)2^{2|\nu|}}{\sigma^{2|\nu|-3}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|-1}\frac{H^2(\eta)}{(-\eta)}\,, \tag{E.10}$$

$$\begin{aligned}\Xi^{(\eta)}_{\pi,\pi} =& \frac{1}{(2\pi)^3}\frac{\Gamma^2(|\nu|)2^{2|\nu|}}{\sigma^{2|\nu|-3}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|-7}\\ &\times\left[\left(\frac{2}{2\nu-1}\right)(|\nu|-\nu)-4\pi G\frac{\dot{\varphi}^2}{H^2(\eta)}\right]^2\frac{1}{H^2(\eta)(-\eta)^7}\,,\end{aligned} \tag{E.11}$$

$$\begin{aligned}\Xi^{(\eta)}_{\varphi,\pi} = \Xi^{(\eta)}_{\pi,\varphi} =& \frac{1}{(2\pi)^3}\frac{\Gamma^2(|\nu|)2^{2|\nu|}}{\sigma^{2|\nu|-3}}\left(\frac{2}{2\nu-1}\right)^{2|\nu|-4}\\ &\times\left[\left(\frac{2}{2\nu-1}\right)(|\nu|-\nu)-4\pi G\frac{\dot{\varphi}^2}{H^2(\eta)}\right]\frac{1}{(-\eta)^4}\,.\end{aligned} \tag{E.12}$$

F Fourier transform on a finite domain

This Appendix is dedicated to show explicitly the solution given by (5.46), while pointing out that the final result depends on a conventional factor. From (5.45), we easily get

$$\sigma_x^2(N) = \sigma_x^2(N_\star)e^{2m(N-N_\star)} + 2\int_{N_\star}^{N} dS\, e^{2m(N-S)} \left\langle \hat{\xi}_x(S)\left(\bar{x}(S) - x_c\right)\right\rangle . \tag{F.1}$$

We can reexpress the expectation value in the second term using (5.43) to get

$$\left\langle \hat{\xi}_x(S)\left(\bar{x}(S) - x_c\right)\right\rangle = \left\langle \int_{S_\star}^{S} e^{m(S-U)}\hat{\xi}_x(S)\hat{\xi}_x(U)dU \right\rangle . \tag{F.2}$$

Using the Fubini theorem, we get

$$\left\langle \hat{\xi}_x(S)\left(\bar{x}(S) - x_c\right)\right\rangle = \int_{S_\star}^{S} e^{m(S-U)} \left\langle \hat{\xi}_x(S)\hat{\xi}_x(U)\right\rangle dU . \tag{F.3}$$

The variance (F.1) is then

$$\sigma_x^2(N) = \sigma_x^2(N_\star)e^{2m(N-N_\star)} + 2\int_{N_\star}^{N}\int_{S_\star}^{S} dSdU\, e^{m(2N-S-U)} \left\langle \hat{\xi}_x(S)\hat{\xi}_x(U)\right\rangle . \tag{F.4}$$

The resolution of (F.1) leads us to consider the following integral:

$$\int_{S_\star}^{S} dU e^{m(2N-S-U)}\delta\left(S-U\right) . \tag{F.5}$$

For a general function, we have

$$\begin{aligned}\int_a^b f(x)\delta(x-x')dx &= \int_{-\infty}^{\infty} dx\, f(x)\left[\theta(x-a) - \theta(x-b)\right]\delta(x-x') \\ &= f(x')\left[\theta(x'-a) - \theta(x'-b)\right] ,\end{aligned} \tag{F.6}$$

what gives in our case

$$\int_{S_\star}^{S} dU e^{m(2N-S-U)} \delta\left(S-U\right) = e^{2m(N-S)}\left[\theta(S-S_\star)-\theta(S-S)\right]$$
$$= e^{2m(N-S)}\left[\theta(S-S_\star)-\theta(0)\right] \ . \tag{F.7}$$

Using the half-maximum convention for the unit step function, we obtain

$$\int_{S_\star}^{S} dU e^{m(2N-S-U)} \delta\left(S-U\right) = \frac{1}{2} e^{2m(N-S)} \ . \tag{F.8}$$

Now, we are able to write the full solution for the variance in x as

$$\sigma_x^2(N) = \sigma_x^2(N_\star) e^{2m(N-N_\star)} + \int_{N_\star}^{N} dS\, e^{2m(N-S)} \Xi_{xx}(S) \ . \tag{F.9}$$

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