Universidade Federal do Espírito Santo

Centro de Ciências Exatas

Programa de Pós-Graduação em Física

Scalar-Tensor Theories of Gravity and Their Cosmological Applications

Olesya Galkina

Vitória

2020

Universidade Federal do Espírito Santo

Centro de Ciências Exatas

Programa de Pós-Graduação em Física

Olesya Galkina

Scalar-Tensor Theories of Gravity and Their Cosmological Applications

Tese apresentada ao Programa de Pós-Graduação em Física da Universidade Federal do Espírito Santo como requisito parcial à obtenção do título de Doutor em Física.

Advisor: Prof. Dr. Júlio C. Fabris

Vitória

2020

Abstract

The main subjects of the thesis concern the most known scalar-tensor theory - Brans-Dicke theory, and cosmological models considered in the gravity in the presence of Born-Infeld type scalar fields, which in the present times are often called tachyons. We introduce the general solutions for the scale factor and the scalar field for the Friedmann–Lemaître–Robertson–Walker flat universe in Brans-Dicke theory. It is usually expected that the violation of the energy conditions is required in order to have classical bounce solutions, even in the nonminimal coupling case: in this situation, phantom fields would appear in the Einstein frame. We show that for the case of the radiative fluid in Brans–Dicke theory it is possible to obtain nonsingular solutions preserving the energy conditions even in the Einstein frame. Then we present a particular solution of Brans-Dicke theory with matter content described by a stiff matter barotropic perfect fluid. It is argued in the literature if Brans–Dicke theory reduces to general relativity in the limit $\omega \to \infty$ if the scalar field goes as $\phi \propto 1/\omega$. We show that the power of time dependence of our particular solution for stiff matter does not depend on ω, and there is no general relativity limit even though we have $\phi \propto 1/\omega$. Finally, we present some cosmological models considered in the gravity in the presence of Born-Infeld type scalar fields, revealing the big brake and other sudden future singularities and the effects of transformations of matter fields. Also, we consider a particular cosmological model describing the smooth transformation between the standard and phantom scalar fields. In both cases we try to find whether is it possible to conserve some kind of notion of particle corresponding to a chosen quantum field present in the universe when the latter approaches the singularity.

Keywords: Brans-Dicke theory, modified gravity, bouncing models, tachyon fields, sudden future singularities.

Resumo

Os principais temas desta tese concernem a teoria escalar-tensorial mais conhecida - a teoria de Brans-Dicke, e os modelos cosmológicos considerados na gravidade na presença de campos escalares do tipo Born-Infeld, que nos tempos atuais são frequentemente chamados de táquions. Começamos com apresentação das soluções gerais do fator de escala e do campo escalar para o universo plano de Friedmann – Lemaître – Robertson – Walker na teoria de Brans-Dicke obtidas por Gurevich et al. Normalmente, espera-se que a violação das condições de energia seja necessária para se ter soluções de ricochete clássicas, mesmo no caso de acoplamento não mínimo: nesta situação, campos fantasmas apareceriam no referencial de Einstein. Mostramos que para o caso do fluido radiativo na teoria de Brans-Dicke é possível obter soluções não singulares preservando as condições de energia mesmo no referencial de Einstein. Em seguida, discutimos o caso de um universo homogêneo e isotrópico descrito por matéria rígida. Geralmente, a teoria de Brans-Dicke se reduz à relatividade geral no limite $\omega \to \infty$ se o campo escalar se comporta como $\phi \propto 1/\omega$. Mostramos que a dependência da potência do tempo da nossa solução particular para matéria rígida não depende de ω, e não há limite de relatividade geral, embora tenhamos $\phi \propto 1/\omega$. Por fim, apresentamos alguns modelos baseados em campos de táquions, revelando "o grande freio" e outras singularidades futuras repentinas e os efeitos das transformações dos campos de matéria. Além disso, consideramos um modelo cosmológico particular que descreve a transformação suave entre os campos escalares padrão e fantasma. Em ambos os casos, tentamos descobrir se é possível conservar algum tipo de noção de partícula correspondente a um campo quântico escolhido quando este se aproxima a singularidade.

Palavras-chave: theoria de Brans-Dicke, gravidade modificada, modelos de ricochete, campos de táquions, singularidades futuras repentinas.

Посвящается моей семье: маме, папе, Ксюше и Варваре.

Acknowledgments

First of all, I would like to express my sincere gratitude to my advisor Prof. Júlio Fabris, who believed in me and accepted as his student. Finishing this PhD would not be possible without his help, patience, motivation, and immense knowledge.

I am very grateful to Prof. Alexander Kamenshchik for all his support during my stay at the University of Bologna and letting me be part of his research group.

Also I would like to thank all my friends and colleagues that I met in Brazil and in Italy who have been supporting me since the very beginning.

This study was financed by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brazil (CAPES) - Finance Code 001.

Contents

List of Figures

Chapter 1

Introduction

General relativity is a highly successful theory to describe gravitational interaction. It is theoretically consistent and experimentally tested theory. By now, general relativity has been passed every experiment in the solar system, for example, the perihelion precession of Mercury or the deflection of light by the Sun [1]-[4]. Moreover, the theory has been confirmed by astrophysical phenomena such as the emission of gravitational waves by binary systems and it is in accordance with the bounds on the velocity of gravitational waves [5, 6]. However, there are strong indications that the theory is incomplete. The discovery of the phenomenon of the cosmic acceleration was the starting point for the formulation of cosmological models containing dark energy, which, due to its specific properties, has been considered responsible for the accelerated expansion of the Universe [7, 8, 9]. Another reason to consider alternative theories of gravity is the initial singularity which existed before the Big Bang. In recent years, there has been increasing interest in cosmological models that replace an initial singularity with a bounce - a smooth transition from contraction to expansion - in order to solve fundamental problems in cosmology. Bouncing models suggest that the universe is eternal, and physical details of a bouncing phase are governed by quantum cosmology.

Brans–Dicke theory is one of the most known modifications of general relativity. It is a scalar-tensor theory which was introduced by Brans and Dicke as a possible implementation of Mach's principle in a relativistic theory. The first part of the thesis is dedicated to some aspects of this theory. In the Section 2 we introduce the classical equations of motion in Brans-Dicke theory and display the general solutions for a scale factor and a scalar field in the case of a flat Friedmann – Lemaître – Robertson – Walker (FLRW) universe.

In the Section 3 we introduce the basic concepts of bouncing cosmology. In order to have a bouncing solution in classical general relativity, violation of the energy conditions is required. A consequence of violating the null energy condition is the appearance of exotic kind of matter fields, for example, a scalar field with negative energy density (ghosts). In the Section 4 we describe a bouncing model in the Brans-Dicke theory with fluids that obey the energy conditions and with no ghosts.

In the Section 5 we analyze a very particular solution for stiff matter in Brans-Dicke theory. We show that the scalar field is inversely proportional to the Brans-Dicke parameter ω. This condition is commonly understood as a sufficient condition for a well-defined general relativity limit. However, we prove that this is not the case for our particular power-law solution.

The second part of the thesis is dedicated to soft future singularities and particles. In the Sections 6 and 7 we describe the concept of sudden future singularities and the survival of the notion of particles approaching the singularity. We show that when approaching the big rip singularity, particles corresponding to the phantom scalar field driving the evolution of the universe must vanish.

In the Section 8 we study cosmological models based on tachyon fields, one of the possible candidates for the role of dark energy, revealing the big brake and other soft future singularities and the effects of transformations of matter fields. In the Section 9 we consider a particular cosmological model describing the smooth transformation between the standard and phantom scalar fields. In both cases we try to find if is it possible to conserve some kind of notion of particle corresponding to a chosen quantum field present in the universe when the latter approaches the singularity.

Finally, in the last part of the thesis we present some concluding remarks.

This thesis consists of works done at the Universidade Federal do Espírito Santo under the supervision of Prof. Júlio Fabris and at the Università di Bologna in collaboration with Prof. Alexander Kamenshchik.

Chapter 2

Scalar-Tensor theories of gravitation and Brans-Dicke theory

Einstein's general theory of relativity (GR) is a geometrical theory of space-time, and the fundamental building block is a metric tensor field. For this reason the theory may be called a "tensor theory." The action that defines the dynamics of gravity in GR is called the Einstein–Hilbert action (also referred to as Hilbert action), and is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R, \tag{2.1}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor matrix, R is the scalar curvature and G is the gravitational constant. The action (2.1) yields the Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor with the Ricci curvature tensor $R_{\mu\nu}$ and $T_{\mu\nu}$ is the stress–energy tensor.

General relativity is a very successful theory to describe gravitational interaction. It is theoretically consistent and experimentally tested theory. So far, GR has been confirmed by every experiment in the solar system [1, 2, 3, 4] and astrophysical phenomena such as the emission of gravitational waves by binary systems and it is in accordance with the bounds on the velocity of gravitational waves [5, 6]. However, there are at least three reasons to seriously consider alternative theories of gravity. The necessity of the dark energy and dark matter for the Standard cosmological model; the theoretical motivation to unify gravity with three other fundamental interactions in a single theoretical framework; and the epistemological fact that alternative theories can be used to highlight the intrinsic properties of GR by showing how it could be otherwise.

One of alternative theories of gravity is "scalar–tensor theory". This type of theory does not merely combine the two kinds of fields. It is built on the solid foundation of GR, and the scalar field comes into play in a highly nontrivial manner, specifically through a "nonminimal coupling term." The prototype of alternative theory of gravity is Brans–Dicke theory (BD). Historically, it is one of the most important alternative to the standard GR theory, which was introduced by Brans and Dicke [10] as a possible implementation of Mach's principle in a relativistic theory. The Brans-Dicke theory can be seen as the first example of Galileons and Horndesky-type theories [11].

Today, a distinguishing feature of modern scalar-tensor theories is that the gravitational coupling is time dependent. This idea belongs to Dirac: his choice was to let the gravitational coupling G become time-dependent, while other fundamental constants remain fixed [12]. In the following decade P. Jordan developed this idea and promoted G to the role of a gravitational scalar field [13]. He presented a general Lagrangian for the scalar field living in four-dimensional curved space-time

$$\mathcal{L}_J = \sqrt{-g}\left\{\phi_J^\gamma\left(R - \frac{\omega_J}{\phi_J^2}g^{\mu\nu}\partial_\mu\phi_J\partial_\nu\phi_J\right) + L_{matter}\left(\phi_J, \Psi\right)\right\}, \tag{2.3}$$

where ϕ_J is Jordan's scalar field, γ and ω_J are constants and Ψ represents matter fields collectively. The introduction of the nonminimal coupling term $\phi_J^\gamma R$ marked the birth of the scalar–tensor theory. The term $L_{matter}\left(\phi_J, \Psi\right)$ was for the matter Lagrangian, which depends generally on the scalar field, as well.

Jordan's idea of a scalar-tensor theory was continued in the work of Brans and Dicke. They defined their scalar field ϕ by

$$\phi = \phi_J^\gamma, \tag{2.4}$$

which simplifies (2.3) since the specific choice of the value of γ is irrelevant. In this way they proposed the basic Lagrangian

$$\mathcal{L}_{BD} = \frac{1}{16\pi}\sqrt{-g}\left\{\phi R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\} + \sqrt{-g}L_{matter}. \tag{2.5}$$

Solar System time-delay experiments set a lower bound on the absolute value of the dimensionless parameter $|\omega| > 500$ [1, 2], which means that BD is strongly constrained for the solar system dynamics. The theory is also constrained by the CMB, as pointed out in Refs. [14] and [15]. However, extensions of BD theory leave place for a varying coupling parameter ω. The Horndeski class of theories cover all possibilities without Ostrogradsky instabilities including the BD theory in its traditional form. This opens the possibility for small values of the coupling parameter in the past (which can be even negative), evolving to a huge value today. Also, the low energy effective action of string theory leads to BD theory with $\omega = -1$ [16]. Brane configurations may allow even more negative values of ω. In evoking this connection, we have mainly in mind the domain of application of the string effective theory which is the primordial Universe.

It is commonly understood that in the $\omega \to \infty$ limit of BD parameter BD theory becomes identical to general relativity [1, 74]. It is true in most situations, but the statement is not valid in general. The crucial point behind this argument is that when the parameter $\omega \gg 1$, the field equations seem to show that $\Box\phi = \mathcal{O}\left(\frac{1}{\omega}\right)$ and hence

$$\phi = \frac{1}{G_N} + \mathcal{O}\left(\frac{1}{\omega}\right), \tag{2.6}$$

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \mathcal{O}\left(\frac{1}{\omega}\right), \tag{2.7}$$

where G_N is Newton's gravitational constant and $G_{\mu\nu}$ is the Einstein tensor. However, there are some examples [75]-[88] where exact solutions cannot be continuously deformed into the corresponding a GR solutions by taking the $|\omega| \to \infty$ limit. Their asymptotic behavior differs exactly because these solutions do not decay as Eq. (2.6) but instead behave as

$$\phi = \frac{1}{G_N} + \mathcal{O}\left(\frac{1}{\sqrt{\omega}}\right). \tag{2.8}$$

Our particular solution, to be developed in the Section 5, has the novelty of having the appropriate asymptotic behavior given by Eq. (2.6) but no GR limit.

Notwithstanding, there are phenomenological applications for Brans-Dicke in cosmology and indeed it has received recently much attention of the scientific community [17]-[26].

2.1 The classical equations of motion

In Brans–Dicke theory the scalar field is understood as part of the geometrical degrees of freedom. This theory has a nonminimal coupling between gravity and the scalar field. The action reads

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g_{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + \int d^4x \sqrt{-g} S_m, \tag{2.9}$$

where S_m is the matter term and ω is the scalar field coupling constant. The variation of the first and second term in the action (2.9) with respect to the metric gives us respectively

$$\delta \left(\int_{R^4} R\phi \sqrt{-g} \right) = \int_{R^4} \left(G_{\mu\nu}\phi + g_{\mu\nu} \Box \phi - \nabla_\mu \nabla_\nu \phi \right) \delta g^{\mu\nu} \sqrt{-g}, \tag{2.10}$$

$$\delta \left(\int_{R^4} \frac{\omega}{\phi} \left(\nabla \phi \right)^2 \sqrt{-g} \right) = \int_{R^4} \left(\frac{\omega}{\phi} \partial_\alpha \phi \partial_\beta \phi - \frac{\omega}{2} g_{\mu\nu} \frac{\nabla_\alpha \phi \nabla^\alpha \phi}{\phi} \right) \delta g^{\mu\nu} \sqrt{-g}, \tag{2.11}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. The variation of the matter term with respect to the metric is

$$\delta \left(\sqrt{-g} \mathcal{L}_m \right) = -\frac{1}{2} \sqrt{-g} T_{\mu\nu}, \tag{2.12}$$

where $T_{\mu\nu}$ is the stress-energy tensor.

Putting all together we have the first field equation

$$G_{\mu\nu} = \frac{1}{\phi} \left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box \phi \right) + \frac{\omega}{\phi^2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) + \frac{8\pi}{\phi} T_{\mu\nu}. \tag{2.13}$$

The variation of the action (2.9) with respect to the scalar field ϕ is

$$\Box \phi = -\frac{\phi}{2\omega} R + \frac{\nabla^\mu \phi \nabla_\mu \phi}{\phi}. \tag{2.14}$$

We rewrite this field equation in the different way. First, multiplying the Eq.

(2.13) by $g^{\mu\nu}$ and taking the trace we have

$$R = -\frac{8\pi}{\phi}T + \frac{\omega}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + \frac{3}{\phi}\Box\phi. \tag{2.15}$$

Then substituting Eq.(2.15) into Eq.(2.14) we obtain the second field equation

$$\Box\phi = \frac{8\pi}{3+2\omega}T. \tag{2.16}$$

The action (2.9) is diffeomorphic invariant and since all variables are dynamic fields we have conservation of energy-momentum, i.e.

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.17}$$

We shall consider the matter content described by a perfect fluid such that the energy-momentum tensor is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}, \tag{2.18}$$

and a barotropic equation of state $p = \alpha\rho$ with $-1 \leq \alpha \leq 1$. The equation of state parameter α is bounded from above in order to avoid superluminal speed of sound. In the extreme case, $\alpha = 1$, the speed of sound equals the speed of light, which corresponds to stiff matter. This equation of state was first proposed by Zeldovich as an attempt to describe matter in extremely dense states such as in the very early universe.

We shall restrict our analysis to the FLRW universes where the metric has a preferred foliation given by homogeneous and isotropic spatial sections. In spherical coordinate system for this particular foliation, the line element has the form

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{2.19}$$

where $a(t)$ is a scale factor function and $k = 0, \pm 1$ defines the spatial section curvature. At the early stages of the expansion of the Universe the curvature is not essential, and we can restrict our further analysis to the quasi-euclidian variant of the isotropic model. In this case the field equations reduces to

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi\frac{\rho}{\phi} - 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2, \tag{2.20}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi\frac{p}{\phi} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{\ddot{\phi}}{\phi} - 2\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}, \tag{2.21}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{8\pi}{(3+2\omega)}\left(\rho - 3p\right), \tag{2.22}$$

where a dot denotes differentiation with respect to the cosmic time t. The continuity equation takes form

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0. \tag{2.23}$$

2.2 Conformal transformation

A conformal transformation transforms a metric $g_{\mu\nu}$ into another metric $\tilde{g}_{\mu\nu}$ according to the rule

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2\left(x\right)g_{\mu\nu}, \tag{2.24}$$

where $\Omega^2\left(x\right)$ is an arbitrary function of space-time coordinates x. This is equivalent to the transformation applied to a line element

$$ds^2 \to \tilde{ds}^2 = \Omega^2 ds^2. \tag{2.25}$$

One speaks of moving from one "conformal frame" to another. More specifically, the Lagrangian (2.5) has a nonminimal coupling term, and such conformal frame is called "Jordan frame" (JCF). By using this conformal transformation (2.24) one can always eliminate the nonminimal coupling term, and the same Lagrangian now is re-expressed in terms of the Einstein–Hilbert term. This conformal frame is called "Einstein frame" (ECF). There is some discussion in the literature about the equivalence or non-equivalence of physics in these two frames [27]-[32]. But in the present work we are not going to discuss this matter.

It is convenient to define a function such that

$$f \equiv \ln \Omega, \tag{2.26}$$

$$f_\mu \equiv \frac{\partial_\mu \Omega}{\Omega}. \tag{2.27}$$

The transformation formula for the scalar curvature in terms of f is

$$R = \Omega^2 \left(\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{\mu\nu} f_\mu f_\nu \right), \tag{2.28}$$

where

$$\tilde{\Box} f = \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu f \right) \tag{2.29}$$

is the d'Alembert operator in the new conformal frame. The second term of the Lagrangian (2.5) can be put in the form

$$\sqrt{-g} \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \sqrt{-\tilde{g}} \Omega^2 \frac{\omega}{\phi} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{2.30}$$

By using Eqs. (2.24), (2.28) and (2.30) we have

$$\mathcal{L} = \frac{1}{16\pi} \sqrt{-\tilde{g}} \left\{ \Omega^{-2} \phi \left(\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{\mu\nu} f_\mu f_\nu \right) - \Omega^{-2} \frac{\omega}{\phi} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} \tag{2.31}$$

$$+ \Omega^{-2} \sqrt{-\tilde{g}} \mathcal{L}_m. \tag{2.32}$$

Since Ω is still arbitrary and undetermined, we choose it in such a way that

$$\Omega^{-2} \phi = 1, \tag{2.33}$$

so the first term of the Lagrangian (2.31) containing $\tilde{R}$ becomes precisely the

Einstein–Hilbert term. The second term of (2.31) disappears after integrating by parts. For the third term of (2.31) we have

$$f_\mu = \frac{1}{2}\frac{\partial_\mu \phi}{\phi}. \tag{2.34}$$

Then

$$6\tilde{g}^{\mu\nu} f_\mu f_\nu = \frac{3}{2}\frac{1}{\phi^2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{2.35}$$

Using Eqs. (2.33) and (2.35), the Lagrangian (2.31) reads

$$\mathcal{L} = \sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{16\pi} - \frac{1}{2}\Delta\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{\phi}\mathcal{L}_m\right\}, \tag{2.36}$$

where

$$\Delta = \left(\frac{2\omega+3}{16\pi}\right)\frac{1}{\phi^2}.$$

Assuming $\Delta > 0$ ($\omega > -\frac{3}{2}$) we introduce a function $\tilde{\phi}$ such that

$$\frac{d\tilde{\phi}}{d\phi} = \sqrt{\Delta}. \tag{2.37}$$

Then

$$\sqrt{\Delta}\partial_\mu\phi = \frac{d\tilde{\phi}}{d\phi}\partial_\mu\phi = \partial_\mu\tilde{\phi}. \tag{2.38}$$

In this case the Lagrangian (2.36) takes form

$$\mathcal{L} = \sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} + \frac{1}{\phi}\mathcal{L}_m\right\}, \tag{2.39}$$

where the second term of (2.36) is now re-expressed as a canonical kinetic term of the new scalar field $\tilde{\phi}$. The relation between ϕ and $\tilde{\phi}$ can be obtained from the differential equation (2.37). Integrating this equation we find

$$\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi}} \ln\left(\frac{\phi}{\phi_0}\right). \tag{2.40}$$

Finally, the action for the ECF is given by

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} + \exp\left(8\sqrt{\frac{\pi}{2\omega+3}}\tilde{\phi}\right)\mathcal{L}_m\right\}. \tag{2.41}$$

In the Jordan frame, the gravitational field is described by the metric tensor $g_{\mu\nu}$ and by the BD field. In the Einstein frame, the gravitational field is described only by the metric tensor $\tilde{g}_{\mu\nu}$, but the scalar field is always present, a reminiscence of its fundamental role in the "old" frame. In addition, the rest of the matter part of the Lagrangian is multiplied by an exponential factor, thus displaying an anomalous coupling to the scalar field $\tilde{\phi}$ [33].

The conformal transformation of the BD action is sometimes seen as a possibility to restrict the range of values of the Jordan frame parameter to $\omega > -\frac{3}{2}$ [34].

2.3 Gurevich's families of solutions

In this section, we present the general solutions for a scale factor and a scalar field in the FLRW flat case in the BD theory obtained by Gurevich *et al* [35]. We shall follow closely their presentation but adapting specifically for the stiff matter case in the section 5.

Substituting the equation of state $p = \alpha\rho$ into the Eq. (2.23) and solving it we obtain

$$\rho = \rho_0 a^{-3(1+\alpha)}, \tag{2.42}$$

where ρ_0 is a constant of integration. Rewrite the Eq. (2.22) as

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3\dot{\phi}\right) = \frac{8\pi}{(3+2\omega)}\left(\rho - 3p\right). \tag{2.43}$$

Using the equation of state $p = \alpha\rho$ we have

$$\rho - 3p = \rho\left(1 - 3\alpha\right). \tag{2.44}$$

Substituting the Eq. (2.44) into the Eq. (2.43) gives us

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3\frac{d\phi}{dt}\right) = \frac{8\pi\rho_0}{(3+2\omega)}\left(1 - 3\alpha\right). \tag{2.45}$$

Let us introduce the new parameterized time θ such that

$$dt = a^{3\alpha}d\theta. \tag{2.46}$$

We have

$$\frac{d\phi}{dt} = \frac{d\phi}{d\theta}\frac{d\theta}{dt} = \frac{d\phi}{d\theta}a^{-3\alpha}. \tag{2.47}$$

Then the equation (2.45) takes form

$$\frac{d}{d\theta}\left(a^{3(1-\alpha)}\frac{d\phi}{d\theta}\right) = \frac{8\pi\rho_0}{(3+2\omega)}\left(1 - 3\alpha\right). \tag{2.48}$$

This equation admits the integral

$$a^{3(1-\alpha)}\frac{d\phi}{d\theta} = \frac{8\pi\rho_0}{(3+2\omega)}\left(1 - 3\alpha\right)\theta + C_1. \tag{2.49}$$

Assume $C_1 = \theta_1\frac{8\pi\rho_0}{(3+2\omega)}\left(1 - 3\alpha\right)$. Then

$$a^{3(1-\alpha)}\frac{d\phi}{d\theta} = \frac{8\pi\rho_0}{(3+2\omega)}\left(1 - 3\alpha\right)\left(\theta + \theta_1\right). \tag{2.50}$$

Rewrite the Eq. (2.21) in the form

$$\frac{1}{a^3}\frac{d}{dt}\left(\dot{a}^3\phi\right) = 3\left\{\omega\left(\rho - p\right) + p\right\}\left(\frac{8\pi}{3+2\omega}\right). \tag{2.51}$$

Similarly, this equation admits the integral

$$\phi a^{-\alpha}\left(\frac{da}{d\theta}\right)^3 = 3\rho_0\sigma\left(\frac{8\pi}{3+2\omega}\right)\left(\theta+\theta_2\right), \tag{2.52}$$

where $\sigma = \omega\left(1-\alpha\right)+1$ and η_2 is an integration constant. Denoting $a^3 = V$ and combining equations (2.50) and (2.52) we obtain the equation for the volume factor

$$\frac{V''}{V} + \left\{\frac{\left(1-3\alpha\right)\left(\theta+\theta_1\right)}{3\sigma\left(\theta+\theta_2\right)} - \alpha\right\}\left(\frac{V'}{V}\right)^2 - \frac{1}{\theta+\theta_2}\frac{V'}{V} = 0, \tag{2.53}$$

where prime denotes the differentiation with respect to θ. The integration of the Eq. (2.53) gives us the expression for the Hubble parameter

$$\frac{a'}{a} = \frac{\sigma\left(\theta+\theta_2\right)}{A\theta^2+B\theta+C}, \tag{2.54}$$

and for the scalar field

$$\frac{\phi'}{\phi} = \frac{\left(1-3\alpha\right)\left(\theta+\theta_2\right)}{A\theta^2+B\theta+C}, \tag{2.55}$$

where $2A = 2\left(2-3\alpha\right)+3\omega\left(1-\alpha\right)^2$, $B = 3\sigma\left(1-\alpha\right)\theta_2+\left(1-3\alpha\right)\theta_1$ and C is an integration constant.

There are three families of solutions of Eqs. (2.54) and (2.55) depending on the value of $\Delta \equiv \tilde{B}^2 - 4A\tilde{C}$, where $\tilde{B} = \frac{B}{\theta_2}$ and $\tilde{C} = \frac{C}{\theta_2}$ are new dimensionless quantities with $\theta_2 \neq 0$. Using the time–time component of Eq. (2.13) and the time component of the conservation of the energy–momentum Eq.

(2.17), one can show that for $\alpha \neq \frac{1}{3}$ the Δ can be recast as

$$\Delta = \frac{(1-3\alpha)^2 \sigma^2}{1+\frac{2}{3}\omega} (\theta_1 - 1)^2 . \tag{2.56}$$

Note that for $\theta_1 \neq 1$ the sign of Δ is negative for $\omega < -\frac{3}{2}$ and positive for $\omega > -\frac{3}{2}$.

The first family of solutions is given by $\Delta < 0$ $\left(\omega < -\frac{3}{2}\right)$. The general solutions for the scale factor and scalar field ϕ read

$$a = a_0 \left[(\theta + \theta_-)^2 + \theta_+^2 \right]^{\sigma/2A} e^{\pm\sqrt{\frac{2}{3}|\omega|-1} f(\theta)}, \tag{2.57}$$

$$\phi = \phi_0 \left[(\theta + \theta_-)^2 + \theta_+^2 \right]^{(1-3\alpha)/2A} e^{\mp 3(1-\alpha)\sqrt{\frac{2}{3}|\omega|-1} f(\theta)}, \tag{2.58}$$

where

$$f(\theta) = \frac{1}{A} \arctan\left(\frac{\theta + \theta_-}{\theta_+} \right), \ \theta_+ = \frac{\sqrt{\Delta}}{2A}, \ \theta_- = \frac{\tilde{B}}{2A}.$$

When the time parameter θ tends to infinity, the scale factor a does not vanish. In this case the infinite contraction occurs till a regular minimum a_{min} and after it is followed by the expansion. Thus, this model with $\omega < -\frac{3}{2}$ admits a cosmological bounce.

The second family is described by $\Delta > 0$ $(\omega > -\frac{3}{2})$. The general solutions are given by

$$a = a_0 (\theta - \theta_+)^{\omega/3\Sigma_\mp} (\theta - \theta_-)^{\omega/3\Sigma_\pm} , \tag{2.59}$$

$$\phi = \phi_0 (\theta - \theta_+)^{\left(1 \mp \sqrt{1+2\omega/3}\right)/\Sigma\mp} (\theta - \theta_-)^{\left(1 \pm \sqrt{1+2\omega/3}\right)/\Sigma\pm} , \tag{2.60}$$

$$\Sigma_\pm = \sigma \pm \sqrt{1 + \frac{2}{3}\omega},$$

where now $\theta_\pm = \left(-\tilde{B} \pm \sqrt{\Delta}\right)/2A$ and $\theta_+ > \theta_-$. This solution always admits a singularity when the scale factor a becomes zero and the density of matter becomes infinite. In the case of $\omega \geq 0$ the initial singularity occurs at $\theta = \theta_+$ for $\alpha \leq 1$ (the "slow" branch of the solution), and for $\alpha < \left(\omega + 1 - \sqrt{1 + \frac{2}{3}\omega}\right)/\omega$ (the "fast" branch of the solution).

There is a third family of solutions given by $\Delta = 0$ ($\theta_1 = 0$) that describes power-law solutions

$$a = a_0 \left(\theta/\theta_0\right)^{\sigma/A} = a_0 \left(t/t_0\right)^{\sigma/A_*}, \tag{2.61}$$

$$\phi = \phi_0 \left(\theta/\theta_0\right)^{\sigma/A} = \phi_0 \left(t/t_0\right)^{(1-3\alpha)/A_*}, \tag{2.62}$$

$$2A_* = 4 + 3\omega\left(1 - \alpha^2\right), \tag{2.63}$$

where we used the relation (2.46) between the parametric time θ with the coordinate time t. These solutions have singular behavior for $t \to 0$ if the power of the scale factor is positive. This happens when $\omega < -1/(1-\alpha)$ for $\alpha \in \left[-1, \frac{1}{3}\right]$ and $\omega < -4/3(1-\alpha)$ for $\alpha \in \left[\frac{1}{3}, 1\right]$.

The third family of solutions can be derived by taking the appropriate limit from the previous two families. Indeed, for $\Delta < 0$ the limit $\Delta \to 0$ implies $\theta_+ \to 0$ with $\theta_- = 0$, hence the Eq. (2.57) shows that $a \propto (\theta + \theta_-)^{\sigma/A}$. For $\Delta > 0$ the limit $\Delta \to 0$ implies $\theta_+ = \theta_- = -B/2A$ and again the Eq. (2.59) gives the same result $a \propto (\theta - \theta_-)^{\sigma/A}$ compatible with the Eq. (2.61).

One of the interesting asymptotic limits for all these solutions is given by finite time but allowing the BD parameter to increase boundlessly. The limit $\omega \to \infty$ depends crucially if $\alpha = 1$ or not. For $\alpha = 1$, the limit $|\omega| \to \infty$ gives $\sigma = \omega(1-\alpha)$, $A = A_* = \frac{3}{2}\omega(1-\alpha)^2$, hence all three functions diverge such that $\phi \to \phi_0$ in all three cases. Furthermore, the scale factor becomes

$$\lim_{\omega \to -\infty} a = a_0 \left[(\theta + \theta_-)^2 + \theta_+^2\right]^{1/3(1-\alpha)}, \text{ for } \theta_1 \neq 1 \text{ and } \omega < -\frac{3}{2}, \tag{2.64}$$

$$\lim_{\omega \to \infty} a = a_0 \left[(\theta - \theta_+)(\theta - \theta_-)\right]^{1/3(1-\alpha)}, \text{ for } \theta_1 \neq 1 \text{ and } \omega > -\frac{3}{2}, \tag{2.65}$$

$$\lim_{|\omega| \to \infty} a = a_0 \left(\theta/\theta_0\right)^{2/3(1-\alpha)} = a_0 \left(t/t_0\right)^{2/3(1-\alpha)}, \text{ for } \theta_1 = 1 \text{ and } \omega \neq -\frac{3}{2}. \tag{2.66}$$

Therefore, if $\alpha \neq 1$, independently of the sign(ω), all three families of solutions asymptotically approach GR. However, that is not the case for stiff matter. For $\alpha = 1$, in the limit $|\omega| \to \infty$, ϕ does not go to a constant and the scalar field does not go to its GR limit. Indeed, the $\alpha = 1$ case has to be studied separately, which is what we shall analyze in the Section 5.

Chapter 3

Bouncing cosmologies

Current observations show that our universe is extremely flat, homogeneous and isotropic at large distances ranging from galactic scales to the entire visible universe. In particular, the observed value of the curvature density parameter

$$\Omega_K \equiv \frac{-K}{H^2 a^2} \tag{3.1}$$

from the latest Planck data combined with BAO [36] is

$$\Omega_{K0} = 0.0007^{+0.0019}_{-0.0019}, \tag{3.2}$$

where $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ is a Hubble parameter and $a(t)$ is a scale factor. The joint results suggest our Universe is spatially flat at the 68% confidence level. Using Eq. (3.1) we can show that the more in the past we go the closer to zero Ω_K gets [37]. At the Planck time (the farthest we can extrapolate the classical theory), which corresponds roughly to redshift $z = 10^{32}$, we have $\Omega_K < 10^{-60}$. The problem is that if $\Omega_K \sim 10^{-59}$ at the Planck era, then today Ω_{K0} would be ten times larger and in complete disagreement with observations. Thus, very special conditions (so-called "fine tuning") were needed in the early universe to give approximate flatness now. It is known as the *flatness problem*.

Another problem concerning standard cosmology is the *horizon problem*. It arises due to the difficulty in explaining the observed homogeneity of causally disconnected regions of space in the absence of a mechanism that sets the same initial conditions everywhere.

The horizon problem, together with the flatness, homogeneity and isotropy of the universe, are first explained by the hypothesis about the existence of an inflation phase [38, 39, 40], i.e. a period of accelerated expansion taking place during the early stages of our universe (between the big bang and the nucleosynthesis). During the inflation phase, the Hubble parameter H is constant, or almost constant, which implies that the scale factor behaves as

$$a\left(t\right) \propto e^{Ht}. \tag{3.3}$$

Thus, the universe went through an exponentially fast expansion phase that extended very small regions of the space to enormous sizes that were many *e-folds number* bigger than their original sizes. As a consequence, the local amount of inhomogeneity, curvature, and anisotropy was effectively smoothed out and became naturally small. In particular, consider that at the beginning of the inflationary phase the spatial curvature is a relevant fraction of the total energy density content of the universe:

$$\frac{|K|}{a_i^2 H_i^2} = \mathcal{O}\left(1\right). \tag{3.4}$$

Suppose that at the end of inflation the scale factor a has grown of a factor e^N, where the number N is called e-folds number. We have

$$\frac{|K|}{a_f^2 H_f^2} = \frac{|K|}{a_i^2 H_i^2} e^{-2N} \approx e^{-2N}. \tag{3.5}$$

Then today we have

$$|\Omega_{K0}| = \frac{|K|}{H_0^2} = \frac{|K|}{a_f^2 H_f^2}\left(\frac{a_f H_f}{H_0}\right)^2 \approx e^{-2N}\left(\frac{a_f H_f}{H_0}\right)^2. \tag{3.6}$$

We need $\frac{a_f H_f}{H_0} < e^N$ in order to have $|\Omega_{K0}| < 1$. To estimate the ratio $\frac{a_f H_f}{H_0}$, we suppose that inflation ends in the radiation-dominated epoch, i.e.

$$H_f^2 \approx \frac{H_0^2 \Omega_{r0}}{a_f^4}, \tag{3.7}$$

then

$$a_f \approx \Omega_{r0}^{1/4}\sqrt{\frac{H_0}{H_f}}. \tag{3.8}$$

We have

$$e^N > \Omega_{r0}^{1/4}\sqrt{\frac{H_f}{H_0}} = \Omega_{r0}^{1/4}\left(\frac{\rho_f}{\rho_0}\right)^{1/4} = \frac{\rho_f^{1/4}}{0,037\,h\,eV}, \tag{3.9}$$

where ρ_f is the energy scale at which radiation ends, which must be larger than 1 MeV^4 if we do not want to spoil big bang nucleosynthesis. Thus, we obtain $N > 17$. In the Planck scale we have $N > 68$ and at the GUT energy scale we have $N > 62$.

The horizon problem is solved since the separate regions of the Universe that were thought to be causally disconnected used to be much closer together and in causal contact before inflation. Quantitatively this problem can be solved in the same way as the flatness problem [37].

However, the inflation models have some problems from theoretical point of view, for example, the eternal inflation problem which suggests that globally inflation never ends. Inflation ends locally to produce pockets of FLRW universes, but there are always region where quantum fluctuations keep the field at high values of the potential energy. Second, in large field models of inflation, the inflaton has to cross a distance in field space larger than the Planck mass M_{Pl} in natural units. This requires non-renormalizable quantum corrections to the action. Third, inflation does not provide a theory of initial conditions. In addition, low initial entropy of the initial state has to be assumed [41]-[43]. These problems of inflation initiated the search for alternatives.

In recent years, there has been increasing interest in cosmological models that replace an initial singularity with a bounce - a smooth transition from contraction to expansion - in order to solve fundamental problems in cosmology. Bouncing models suggest that the universe is eternal, and physical details of a bouncing phase are governed by quantum cosmology. The bounce models include quantum gravity based models (loop quantum gravity and canonical quantum cosmology), ekpyrotic and cyclic scenarios, a nonsingular bounce in string theory [44]- [46]. The bounces exist in the closed Friedmann models with a scalar fields and they were extensively studied in [55]-[58], [62, 64, 65, 66].

The conditions to have a bounce are

$$\ddot{a}\left(t\right) > 0, \tag{3.10}$$

$$\dot{a}\left(t_0\right) = 0. \tag{3.11}$$

This means that the scalar factor a is a convex function and it has a global minimum at $t = t_0$.

Adding a contraction phase solves the horizon problem since far separated regions of the universe today were in casual contact during the contraction phase. Similarly, the flatness problem together with homogeneity and isotropy of the universe can be solved by having a smoothing mechanism in the contraction phase. In addition, there is no initial singularity problem since bouncing models are geodesically complete cosmological scenarios.

Usually, to obtain a bouncing solution in GR, violation of the null energy condition (NEC) is required [47]:

$$\rho + p \geq 0, \tag{3.12}$$

where ρ is energy density and p is pressure as usual. To show this, consider the Einstein equations

$$H^2 + \frac{K}{a^2} = \frac{1}{3}\rho, \tag{3.13}$$

$$\dot{H} + H^2 = -\frac{1}{6}\left(\rho + 3p\right), \tag{3.14}$$

where dot denotes a derivative with respect to cosmic time t. The constant K defines the spatial curvature and can be equal to $-1, 0, 1$ for an open, flat or closed universe respectively. From the Eqs. (3.13) and (3.14) the time derivative of the Hubble parameter reads

$$\dot{H} = \frac{K}{a^2} - \frac{1}{2}\left(\rho + p\right). \tag{3.15}$$

Thus, in the case of flat spatial sector $(K = 0)$ it is required $\rho + p < 0$ in order to have $\dot{H} > 0$. Note that the condition $\dot{H} > 0$ implies $\ddot{a} > 0$ for a positive a. A consequence of violating the null energy condition is the appearance of exotic kind of matter fields, for example, a scalar field with negative energy density (ghosts). A crucial point in bouncing models is actually to construct a regular model in which such ghosts are absent while still having a bouncing phase. In the section 4 we present the bouncing solution in Brans–Dicke theory with radiative fluid that obeys the energy conditions, and with no ghosts.

Bouncing models without violating NEC may be obtained outside GR, for example, in models with non minimal couplings [48], using Weyl geometries [49], with curvature squared terms [50]-[53], and branes within charged ADS black holes [54].

Chapter 4

Regular bouncing solutions, energy conditions and Brans-Dicke theory

In this chapter we describe a nonsingular model with fluids that obey the energy conditions and with no ghosts. This is possible even in the simplest scalar–tensor theory, the Brans–Dicke theory. We will essentially analyze the solutions found by Gurevich et al. [35] for a flat homogeneous and isotropic Universe. Our goal is to identify some properties of these already known solutions which, to our knowledge, have not been studied in some of their aspects. These properties may be relevant for the construction of a coherent and realistic cosmological model, in particular for solving the singularity problem.

In the Ref. [35] have been determined the general solutions for the cosmological isotropic and homogeneous flat Universe with a perfect fluid with an equation of state $p = \alpha\rho$, where α is a constant such that $0 \leq \alpha \leq 1$. We presented these solutions in details in the section 2.3. Let us recall briefly these solutions.

The general solution for $\omega > -\frac{3}{2}$ (a case where the energy conditions for the scalar field are satisfied) reads

$$a(\theta) = a_0 (\theta - \theta_+)^{r_+} (\theta - \theta_-)^{r_-} , \tag{4.1}$$

$$\phi(\theta) = \phi_0 (\theta - \theta_+)^{s_+} (\theta - \theta_-)^{s_-} , \tag{4.2}$$

with the definitions

$$r_+ = \frac{\omega}{3\left[\sigma \mp \sqrt{1+\frac{2}{3}\omega}\right]}, \quad r_- = \frac{\omega}{3\left[\sigma \pm \sqrt{1+\frac{2}{3}\omega}\right]}, \tag{4.3}$$

$$s_+ = \frac{1 \mp \sqrt{1+\frac{2}{3}\omega}}{\sigma \mp \sqrt{1+\frac{2}{3}\omega}}, \quad s_- = \frac{1 \pm \sqrt{1+\frac{2}{3}\omega}}{\sigma \pm \sqrt{1+\frac{2}{3}\omega}}, \tag{4.4}$$

where $\sigma = 1+\omega(1-\alpha)$, and a_0, ϕ_0, $\theta_\pm$ are arbitrary constants, with $\theta_+ > \theta_-$. The time coordinate θ is connected with the cosmic time t by

$$dt = a^{3\alpha} d\theta. \tag{4.5}$$

For $\omega < -\frac{3}{2}$, where there is violation of the energy conditions for the scalar field in the Einstein frame, as it will be discussed below, the solutions read

$$a = a_0 \Big[(\theta + \theta_-)^2 + \theta_+^2 \Big]^{\big(1+(1-\alpha)\omega\big)/A} e^{\pm 2f(\theta)/A}, \tag{4.6}$$

$$\phi = \phi_0 \left[(\theta + \theta_-)^2 + \theta_+^2 \right]^{(1-3\alpha)/A} e^{\pm 6(1-\alpha)f(\theta)/A}, \tag{4.7}$$

where

$$f(\theta) = \sqrt{\frac{2|\omega| - 3}{3}} \arctan\left(\frac{\theta + \theta_-}{\theta_+}\right) \tag{4.8}$$

$$A = 2(2 - 3\alpha) + 3\omega (1-\alpha)^2 . \tag{4.9}$$

In the case $\omega > -\frac{3}{2}$, the condition to have a regular bounce can be expressed by requiring $r_+ < 0$ (the scale factor is infinite at one asymptote), $r_+ + r_- > 0$ (the scale factor is infinite at another asymptote) and $3\alpha r_+ + 1 < 0$ (the cosmic time varies from $-\infty$ to $+\infty$). These conditions imply that a regular bounce may be obtained for $\frac{1}{4} < \alpha < 1$ and $-\frac{3}{2} < \omega \leq -\frac{4}{3}$. The case $\alpha = 1$ is quite peculiar, and contains no bounce [59].

We will be interested here mainly in a scenario for the early universe. Thus, we will consider in detail the radiative universe. The solution by Gurevich et al for the radiative case ($p = \frac{1}{3}\rho$) is given by the following expressions

- $\omega > -\frac{3}{2}$:

$$a(\eta) = a_0(\eta - \eta_+)^{\frac{1}{2}(1\pm r)}(\eta - \eta_-)^{\frac{1}{2}(1\mp r)}, \tag{4.10}$$

$$\phi(\eta) = \phi_0(\eta - \eta_+)^{\mp r}(\eta - \eta_-)^{\pm r}; \tag{4.11}$$

- $\omega < -\frac{3}{2}$:

$$a(\eta) = a_0[(\eta + \eta_-)^2 + \eta_+^2]^{\frac{1}{2}} e^{\pm\frac{1}{\sqrt{\frac{2}{3}|\omega|-1}} \arctan\frac{\eta+\eta_-}{\eta_+}}, \tag{4.12}$$

$$\phi(\eta) = \phi_0 e^{\mp\frac{2}{\sqrt{\frac{2}{3}|\omega|-1}} \arctan\frac{\eta+\eta_-}{\eta_+}}. \tag{4.13}$$

In these expressions,

$$r = \frac{1}{\sqrt{1 + \frac{2}{3}\omega}}, \tag{4.14}$$

η is the conformal time and $\eta_\pm$ are constants such that $\eta_+ > \eta_-$.

If we perform a conformal transformation of the Brans-Dicke action such that $g_{\mu\nu} = \phi^{-1}\tilde{g}_{\mu\nu}$, we re-express it in the so-called Einstein's frame (see Section 2.2)

$$S = \frac{1}{16\pi}\int d^4x \Big\{ \sqrt{-\tilde{g}}\Big[\tilde{R} - \Big(\omega + \frac{3}{2}\Big)\frac{(\nabla\phi)^2}{\phi^2}\Big] + \mathcal{L}_m \Big\}.$$

Thus, in the Einstein frame, $\omega > -\frac{3}{2}$ corresponds to an ordinary scalar field with positive energy density, while for $\omega < -\frac{3}{2}$, the kinetic term of the scalar field changes sign, and it becomes a phantom field with negative energy density. Remember that the radiative fluid is conformally invariant.

4.1 Analysis of the solutions

For $\omega \geq 0$ the scale factor displays an initial singularity followed by expansion, reaching $a \to \infty$ as $\eta \to \infty$. Note that the radiative universe of GR characterized by

$$a \propto \eta, \tag{4.15}$$

can be recovered from the above solutions if $\eta_\pm = 0$, in the limit $\omega \to \infty$ when $\eta_+ = \eta_-$, or in the asymptotic limit $\eta \to \infty$.

The GR behavior of the scale factor is also achieved for $\omega = 0$. However, in this case, the scalar field (the inverse of the gravitational coupling) varies with time, and its variation depends essentially on the sign in the exponent in Eqs. (4.10)-(4.11). For the upper sign, we find

$$a(\eta) = a_0(\eta - \eta_+), \tag{4.16}$$

$$\phi(\eta) = \phi_0 \frac{\eta - \eta_-}{\eta - \eta_+}, \tag{4.17}$$

and the scalar field decreases monotonically from infinite to a constant (positive) value as the universe evolves. For the lower sign, the behavior of the functions are given by

$$a(\eta) = a_0(\eta - \eta_-), \tag{4.18}$$

$$\phi(\eta) = \phi_0 \frac{\eta - \eta_+}{\eta - \eta_-}, \tag{4.19}$$

and the scalar field increases monotonically from an infinite negative value to a constant positive value as the universe evolves: initially there is a repulsive gravitational phase. This can be considered as a Big Rip type singularity since it occurs when $a \to \infty$ at finite proper time.

Bounce solutions can be obtained from the Gurevich et al solutions in the radiative case if the lower sign is chosen in Eqs. (4.10)-(4.11) for $-\frac{3}{2} < \omega < 0$. However, there is a singularity at $\eta = \eta_+$ for $-\frac{4}{3} < \omega < 0$ at $\eta = \eta_+$, even if the scale factor diverges at this point. On the other hand, if $-\frac{3}{2} < \omega \leq -\frac{4}{3}$, the bounce solutions are always regular, with no curvature singularity. Note that the gravitational coupling diverges, but only at infinite cosmic time, where the scale factor is also infinite. One can expect that instabilities (due to the anisotropic perturbations) do not develop since, in this situation, anisotropies are suppressed as they decay fast when the scale factor increases. This kind of instabilities may be very relevant, however, if there is a change of sign in the gravitational coupling at finite scale factor, as in the case of Ref. [60]. In this last case $\left(-\frac{3}{2} < \omega \leq -\frac{4}{3}\right)$, there are two possible scenarios (thanks to time reversal invariance):

1. A universe that begins at $\eta = \eta_+$ with $a \to \infty$, with an infinite value for the gravitational coupling ($\phi = 0$), evolving to the other asymptotic limit with $a \to \infty$ but with ϕ constant and finite;

2. The inverse behaviour occurs for $-\infty < \eta < -\eta_+$.

In both cases, the cosmic times ranges $-\infty < t < \infty$. The dual solution in the Einstein frame for $-\frac{3}{2} < \omega \leq -\frac{4}{3}$ is given by $b(\eta) = b_0(\eta-\eta_+)^{1/2}(\eta-\eta_-)^{1/2}$ (with $b = \phi^{1/2}a$) and contains an initial singularity. This can be considered as a specific case of "conformal continuation" in the scalar-tensor gravity proposed in [61].

For the special case $\omega = -\frac{4}{3}$ there is still no singularity if we choose the lower sign. In this case, the scale factor and the scalar field behave

$$a(\eta) \propto \frac{(\eta - \eta_-)^2}{\eta - \eta_+}, \quad \phi(\eta) \propto \left(\frac{\eta - \eta_+}{\eta - \eta_-}\right)^3. \tag{4.20}$$

If $-\infty < \eta < \eta_+$ the universe begins with $a \to \infty$, with ϕ constant and finite, while in the remote future $a \to \infty$ and $\phi = 0$. If we choose the interval $\eta_+ \leq \eta < \infty$, the scenario is reversed, and we get the possibility to have a constant gravitational coupling today.

For $\omega = -\frac{4}{3}$ and the upper sign the solutions exhibit an initial singularity:

$$a(\eta) \propto \frac{(\eta - \eta_+)^2}{\eta - \eta_-}, \tag{4.21}$$

$$\phi(\eta) \propto \left(\frac{\eta - \eta_-}{\eta - \eta_+}\right)^3. \tag{4.22}$$

Similar features for the scale factor and the scalar field are reproduced for $\omega < -\frac{3}{2}$. However the scalar field has a phantom behavior as already stated.

4.2 Energy conditions and perturbations

An important aspect of these solutions concerns the energy conditions. In general in order to have a bouncing solution, violation of the energy conditions is required. The strong and null energy conditions in General Relativity are given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0, \tag{4.23}$$

$$-2\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right) = 8\pi G(\rho + p) > 0. \tag{4.24}$$

In order to use the energy condition in this form the Brans-Dicke theory must be reformulated in the Einstein frame. It is easy to verify that both energy conditions are satisfied as far as $\omega < -\frac{3}{2}$. This is consistent with the fact that in the Einstein frame the cosmological scenarios are singular unless $\omega < -\frac{3}{2}$. On the other hand, in the original Jordan frame there are non singular models if $-\frac{3}{2} < \omega < -\frac{4}{3}$. But in this range the scalar field obeys the energy condition. The effects leading to the avoidance of the singularity come from the non-minimal coupling. We plot the "effective" energy condition, represented in the left-hand side of Eqs. (4.23)-(4.24), taking into account the effects of the non-minimal coupling. If we consider only the left-hand side of the relations Eqs. (4.23)-(4.24), the effects of the interaction due to the non-minimal coupling are included, and the energy conditions can be violated even if the matter terms do not violated them. In Fig. 4.1 we show the expressions for these relations for some values of ω.

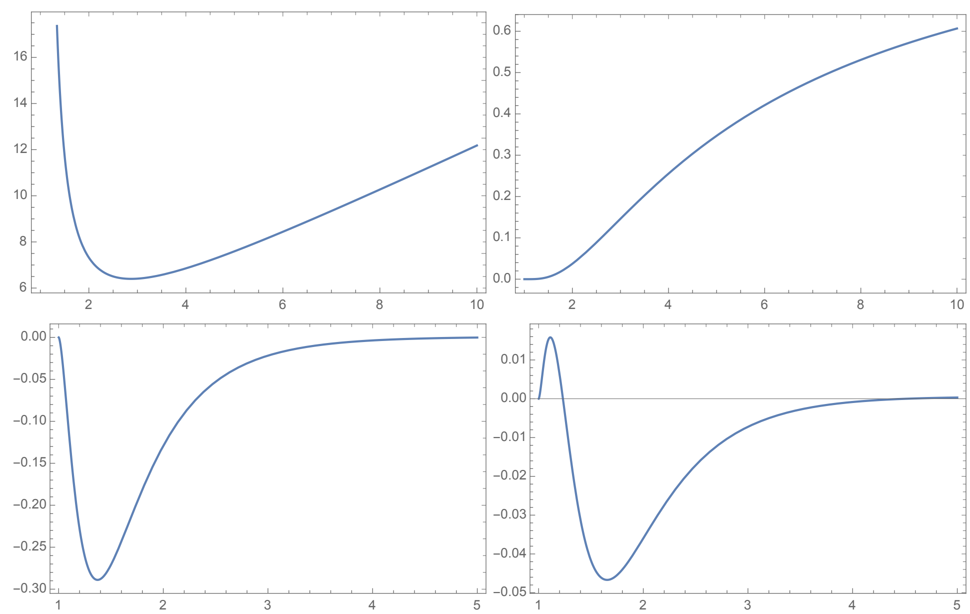

Figure 4.1: Behavior of the (from left to right) scale factor, scalar field, "effective" strong energy condition, and "effective" null energy condition (right) for $\omega = -1.43$.

It is interesting to notice that, for the most usual fluids employed in cosmology, the case of the radiative fluid is the only one where the possibility of obtaining a singularity-free scenario preserving the energy conditions is possible, at least in the Brans-Dicke theory. It is true also for the model with flat spatial sections. For a non-flat universe, a singularity-free scenario can be obtained even in General Relativity if the strong energy condition (but not necessarily the null energy condition) is violated. For analysis of bouncing solutions in closed universe with and without violation of the energy conditions see [62]-[66].

For a dust ($p = 0$), the scale factor can be expressed in terms of the cosmic time and behaves, according to the Gurevich et al solution, as

$$a(t) = a_0(t - t_+)^{r_\pm}(t - t_-)^{r_\mp}, \quad r_\pm = \frac{1 + \omega \pm \sqrt{1 + \frac{2}{3}\omega}}{4 + 3\omega}, \tag{4.25}$$

$t_\pm$ being integration constants such that $t_+ > t_-$. There is a singular bounce for negative values of ω . In their work, Gurevich et al does not display explicitly the solution for a vacuum equation of state ($p = -\rho$) but it can be deduced from a general expression they write down. For $p = -\rho$ the general solution reduces to

$$a(\theta) = a_0(\theta - \theta_+)^{s_\pm}(\theta - \theta_-)^{s_\mp}, \tag{4.26}$$

$$s_\pm = \frac{1 + 2\omega \pm \sqrt{1 + \frac{2}{3}\omega}}{2(5 + 6\omega)}, \tag{4.27}$$

where θ is a parametric time, which is connected to cosmic time through the relation $dt = a^{-3}d\theta$. As in the pressureless matter case, bounce solutions exist for negative ω, but they are singular. Of course, in both pressureless and cosmological constant cases singularity free solutions are possible if $\omega < -\frac{3}{2}$ but this implies a phantom scalar field.

In order to derive the perturbed equations, we decompose the metric tensor as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{4.28}$$

with $|h_{\mu\nu}| \ll 1$. Then the time-time component of the Ricci tensor takes form

$$\tilde{R}_{00} = R_{00} + \delta R_{00}, \tag{4.29}$$

where

$$\delta R_{00} = \frac{1}{a^2}\left(\ddot{h}_{kk} - 2\frac{\dot{a}}{a}\dot{h}_{kk} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)h_{kk}\right). \tag{4.30}$$

The time-time component of the energy-momentum tensor is given by

$$\tilde{T}^{00} = T^{00} + \delta T^{00} = T^{00} + \delta\rho, \tag{4.31}$$

and its trace reads

$$\tilde{T} = T + \delta T = T + \delta\rho - 3\delta p. \tag{4.32}$$

The perturbation of the scalar field is defined as $\delta\phi(x) = \phi(x) - \phi^{(0)}(t)$. The d'Alembertian operator of the scalar field is expressed by

$$\delta\,\Box\,\phi = \delta\ddot{\phi} + a\dot{a}h^{kk}\dot{\phi} - \frac{1}{2a^2}\dot{h}_{kk}\dot{\phi} + 3\frac{\dot{a}}{a}\delta\dot{\phi} - \frac{\nabla^2}{a^2}\delta\phi. \tag{4.33}$$

Using expression above, the equations (2.20) and (2.22) read respectively

$$\frac{\ddot{h}}{2} + H\dot{h} = \frac{8\pi\rho}{\phi}\left(\frac{2+\omega+3\alpha(1+\omega)}{3+2\omega}\right)(\delta-\lambda) + \ddot{\lambda} + 2(1+\omega)\frac{\dot{\phi}}{\phi}\dot{\lambda}, \tag{4.34}$$

$$\ddot{\lambda} + \left(3H + 2\frac{\dot{\phi}}{\phi}\right)\dot{\lambda} + \left[\frac{k^2}{a^2} + \left(\frac{\ddot{\phi}}{\phi} + 3H\frac{\dot{\phi}}{\phi}\right)\right]\lambda = \frac{8\pi(1-3\alpha)\rho}{(3+2\omega)\phi}\delta + \frac{\dot{\phi}}{\phi}\frac{\dot{h}}{2}, \tag{4.35}$$

and the conservation of energy-momentum tensor decompose in two equations

$$2\dot{\delta} - (1+\alpha)\left(\dot{h} - 2U\right) = 0, \tag{4.36}$$

$$(1+\alpha)\left(\dot{U} + (2-3\alpha)HU\right) = \alpha\frac{k^2}{a^2}\delta. \tag{4.37}$$

In these equations we use $\delta = \frac{\delta\rho}{\rho}$, $\lambda = \frac{\delta\phi}{\phi}$, $U = \delta u^i_{,i}$, $h = \frac{h_{kk}}{a^2}$. Moreover, k is the wavenumber coming from the Fourier decomposition and H is the Hubble function.

Particularizing the expressions (4.34)-(4.37) for a radiative fluid $\left(\alpha = \frac{1}{3}\right)$, the perturbed equations read

$$\ddot{h} + 2H\dot{h} = \frac{16\pi}{\phi}\rho(\delta - \lambda) + 2\ddot{\lambda} + 4\frac{\dot{\phi}}{\phi}(1+\omega)\dot{\lambda}, \tag{4.38}$$

$$\ddot{\lambda} + \left(3H + 2\frac{\dot{\phi}}{\phi}\right)\dot{\lambda} + \frac{k^2}{a^2}\lambda = \frac{\dot{\phi}}{\phi}\frac{\dot{h}}{2}, \tag{4.39}$$

$$\dot{\delta} + \frac{4}{3}\left(\theta - \frac{\dot{h}}{2}\right) = 0, \tag{4.40}$$

$$\dot{\theta} + H\theta = \frac{k^2}{4a^2}\delta, \tag{4.41}$$

where $\theta = \partial_i \delta u^i$.

The evolution of scalar perturbations in the Brans-Dicke theory has been studied in Ref. [67], and some features connected with the Gurevich et al solutions have been displayed in Ref. [68]. For the bouncing regular solutions analyzed here, it is natural to implement the Bunch-Davies vacuum state as the initial condition. However, it is known that in bounce scenario a flat or almost flat spectrum requires a matter dominated period in the contraction phase. This is not obviously the case for the regular Gurevich et al solutions which is verified for a radiative fluid.

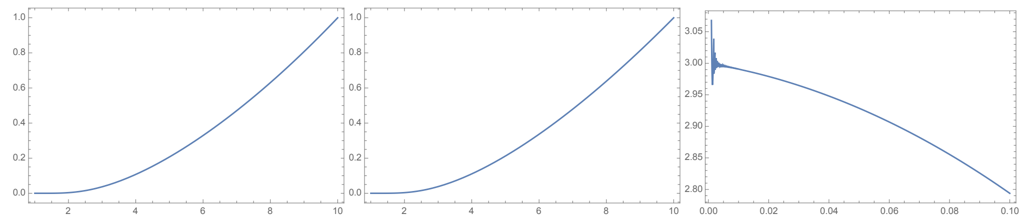

Figure 4.2: Behavior of the density contrast is displayed for $k = 0.01$ (left panel) and for $k = 0.1$ (center panel). The normalization has been chosen such that the final density contrast is equal to one. In the right panel is shown the dependence of the spectral index n_s on the wavenumber k. All the figures were obtained for $\omega = -1.43$ lower sign.

In Fig. 4.2 we display the evolution for the density contrast for $k = 0.01$ and $k = 0.1$ (in the units of the current Hubble scale), as well as the dependence of the spectral index n_s as a function of the wavenumber k. The spectral index is defined as usual

$$\Delta = k^3 \delta_k^2 = k^{n_s - 1}. \tag{4.42}$$

We display the evolution of the perturbations and the dimensionless power spectrum which exhibits a clear disagreement with the observations (compare with similar results obtained in Ref. [69]). Since the model studied here requires a single radiative fluid such somehow negative result could be expected from the beginning.

Chapter 5

Stiff matter solution in Brans–Dicke theory and GR Limit

In this chapter, we display a particular solution of BD with matter content described by a stiff matter barotropic perfect fluid. This is a very interesting solution with exotic characteristics revealing some of the new features, for better or worse, that one can expect to find in BD-like alternative theories of gravity [70, 71, 72, 73]. In particular, the time evolution of the system is independent of the value of the parameter ω. The evolution of the perturbations has only growing modes, which is also another distinct feature of this solution. In addition, the scale factor evolution behaves as $a \propto t^{1/2}$, typical of radiation dominated epoch in GR, hence this configuration might have some applications in the early universe.

It is argued in the literature whether BD approaches GR in the $|\omega| \to \infty$ limit [74]. The crucial point behind this argument is that when the parameter

$\omega \gg 1$, the field equations seem to show that $\Box\phi = \mathcal{O}\left(\frac{1}{\omega}\right)$ and hence

$$\phi = \frac{1}{G_N} + \mathcal{O}\left(\frac{1}{\omega}\right), \tag{5.1}$$

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \mathcal{O}\left(\frac{1}{\omega}\right), \tag{5.2}$$

where G_N is Newton's gravitational constant and $G_{\mu\nu}$ is the Einstein tensor. However, there are some examples [75]-[88] where exact solutions cannot be continuously deformed into the corresponding a GR solutions by taking the $|\omega| \to \infty$ limit. Their asymptotic behavior differs exactly because these solutions do not decay as Eq. (5.1) but instead behave as

$$\phi = \frac{1}{G_N} + \mathcal{O}\left(\frac{1}{\sqrt{\omega}}\right). \tag{5.3}$$

Our particular solution, to be developed in the following subsection, has the novelty of having the appropriate asymptotic behavior given by Eq. (5.1) but no GR limit.

5.1 Exact power-law solution for stiff matter

In the section 2.3 we presented three families of flat space solutions for the equation of state $p = \alpha\rho$ obtained by Gurevich *et al.* In this section we will adapt specifically for the stiff matter case ($\alpha = 1$) and then present our particular power-law solution.

In the case of $\omega < -\frac{3}{2}$ the solutions (2.57) and (2.58) simplify to

$$a = a_0\left[(\theta+\theta_-)^2 + \theta_+^2\right]^{-\frac{1}{2}} \exp\left\{\mp\sqrt{\frac{2|\omega|}{3} - 1}\,\arctan\left(\frac{\theta+\theta_-}{\theta_+}\right)\right\}, \tag{5.4}$$

$$\phi = \phi_0\left[(\theta+\theta_-)^2 + \theta_+^2\right]. \tag{5.5}$$

Note that the scalar field dynamics does not depend on ω anymore and the scale factor has a mild dependence on this parameter. The asymptotic behavior of solutions (5.4) and (5.5) for $\theta \to \pm\infty$ are

$$a(\theta) \propto \exp\left(\epsilon \frac{\pi}{2}\sqrt{\frac{2|\omega|}{3} - 1}\right)\frac{1}{\theta}, \tag{5.6}$$

$$\phi(\theta) \propto \theta^2. \tag{5.7}$$

where $\epsilon = \mp 1$ for $\theta \to -\infty$ and $\epsilon = \pm 1$ for $\theta \to +\infty$. In this asymptotic behavior, the cosmic time goes as $t \propto \theta^{-2}$, hence, the scale factor Eq. (5.6) goes as $a \propto t^{1/2}$ in both asymptotic limits $\theta \to \pm\infty$. Similarly, the asymptotic behavior of the scalar field is $\phi \propto \theta^2 \propto t^{-1}$. The two asymptotic limits $\theta \to \pm\infty$ describe two distinct possible evolutions where the universe behaves as a GR radiation dominated phase. For $\theta \to -\infty$, the universe starts from an initial singularity at $t = 0$ and expands there from with a radiation dominated phase. In the limit $\theta \to +\infty$, the universe contracts from infinity until it reaches a big crunch singularity again at $t = 0$ during a radiation dominated phase.

The second family of solutions ($\omega > -\frac{3}{2}$) is described by equations (2.59)-(2.60) and in the stiff matter case reduces to

$$a = a_0\left(\theta - \theta_+\right)^{\omega/\left[3\left(1 \mp \sqrt{1+2\omega/3}\right)\right]}\left(\theta - \theta_-\right)^{\omega/\left[3\left(1 \pm \sqrt{1+2\omega/3}\right)\right]}, \tag{5.8}$$

$$\phi = \phi_0\left(\theta - \theta_+\right)\left(\theta - \theta_-\right). \tag{5.9}$$

The asymptotic behavior of equations (5.8) and (5.9) shows that for $|\theta| \gg \theta_+$ we have $a \propto \theta^{-1} \propto t^{1/2}$ and $\phi \propto \theta^2 \propto t^{-1}$. Thus, we have the same asymptotic behavior as in the previous case given by equations (5.6) and (5.7).

Let us study the dynamic for FLRW universe filled with a stiff matter perfect fluid. Following Gurevich's et al. family of solutions, we propose a power-law ansatz such that

$$a = a_0 t^r,\ \phi = \phi_0 t^s. \tag{5.10}$$

Equating the power in the Klein-Gordon Eq. (2.22), it is easy to check $6r + s = 2$. Furthermore, the coefficients of Eqs. (2.20) and (2.22) imply

$$3r^2 + 3rs - \frac{\omega}{2}s^2 = \frac{8\pi\rho_0}{\phi_0} - 3\frac{k}{a_0^2}t^{2-2r}, \tag{5.11}$$

$$s(s-1) + 3rs = -\frac{16\pi\rho_0}{(3+2\omega)\,\phi_0}. \tag{5.12}$$

Using the previous relation for r and s and combining these two equations, we find

$$r = \frac{1}{2}, \quad s = -1 \tag{5.13}$$

for spatial curvature $k = 0$, and

$$r = 1, \quad s = -4.$$

for $k \neq 0$. Note also that the equations require the constraint

$$\phi_0 = -\frac{32\pi}{(3+2\omega)}\rho_0 \tag{5.14}$$

for flat spatial sections, and

$$\phi_0 = -\frac{2\pi}{3+2\omega}\rho_0 \tag{5.15}$$

with $k = -a_0^2$ normalized to -1 for non-flat. From now on we will focus solely on the solution (5.13), for a $k = 0$ FLRW universe.

Therefore, in order to have an attractive gravity with positive energy density, i.e. ρ_0 and $\phi_0 > 0$, we must have $\omega < -\frac{3}{2}$. During the whole evolution the scale factor mimics a GR radiation dominated expansion, namely $a \propto$

$t^{1/2}$, while the scalar field is always decreasing inversely proportional to the cosmic time $\phi \propto t^{-1}$. There is no GR limit in the sense that the evolution does not depend on the BD parameter ω. Furthermore, Eq. (5.14) seems to show that the limit $\omega \to \infty$ is not even well defined since the gravitation strength is inversely proportional to the scalar field, i.e $G \sim \phi^{-1}$. Notwithstanding, this power law solution is completely consistent for finite values of ω. In the next section we shall study the cosmological perturbations over this particular solution.

We would like to point out to a possible cosmological realization of our solution (5.13). Alternative theories of gravity, such as BD, are commonly used in cosmology to explain the late time acceleration of our universe. One of these models are Quintessence models, in which the matter component is described by a minimally coupled scalar field, with a potential $V(\psi)$. Some of these models [91], are such that its potential goes to zero at early times, which implies that in this period the scalar field has a stiff matter type equation of state. Therefore, the model described in this work can be interpreted as the early time description of such models.

5.2 Cosmological perturbations

Consider the background solution found above for a flat FLRW universe filled with stiff matter in BD theory, i.e.

$$a = a_0 t^{1/2}, \quad \phi = \phi_0 t^{-1} \tag{5.16}$$

In Ref. [89] the general perturbed equations for a fluid with equation of state of the type $p = \alpha\rho$, with α constant, has been established for the Brans-Dicke cosmology. The full perturbed dynamical system is given by the perturbed version of Eqs. (2.13) and (2.14). However, the evolution of the matter density perturbation can be analyzed using only the perturbed version of the time-time Einstein's equations, the Klein-Gordon, and the conservation of energy-momentum tensor.

The metric perturbation is defined as $h_{\mu\nu} = g_{\mu\nu} - g^{(0)}_{\mu\nu}$, where $g^{(0)}_{\mu\nu}$ is given by the FLRW solution with Eq. (5.16) and $k = 0$. Following Ref. [89], we adopt the synchronous gauge where $h_{0\mu} = 0$. It is straightforward to calculate the perturbed Ricci tensor, which has time-time component given by

$$\delta R_{00} = \frac{1}{a^2}\left[\ddot{h}_{kk} - 2\frac{\dot{a}}{a}\dot{h}_{kk} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)h_{kk}\right]. \tag{5.17}$$

The time-time component of the perturbed energy-momentum tensor and its trace read

$$\delta T^{00} = \delta\rho, \quad \delta T = \delta\rho - 3\delta p. \tag{5.18}$$

Similarly, the perturbation of the scalar field is defined as $\delta\phi\,(x) = \phi\,(x) - \phi^{(0)}\,(t)$. The d'Alembertian of the scalar field is

$$\Box\delta\phi = \delta\ddot{\phi} + a\dot{a}h^{kk}\dot{\phi} - \frac{1}{2a^2}\dot{h}_{kk}\dot{\phi} + 3\frac{\dot{a}}{a}\delta\dot{\phi} - \frac{1}{a^2}\nabla^2\delta\phi. \tag{5.19}$$

It is convenient to define new variables. In particular, we define the usual expression for the density contrast, a similar version for the perturbation of the scalar field, the divergence of the perturbation of the perfect fluid's velocity field (δu^i), and a normalized version of the metric perturbation. They are defined, respectively, as

$$\delta = \frac{\delta\rho}{\rho}, \quad \lambda = \frac{\delta\phi}{\phi}, \quad U = \delta u^i_{,i}, \quad h = \frac{h_{kk}}{a^2}. \tag{5.20}$$

Using (5.20) and decomposing them in Fourier modes n, the time-time BD and the Klein-Gordon, equations (2.20) and (2.22), read respectively

$$\frac{\ddot{h}}{2} + H\dot{h} = \frac{8\pi\rho}{\phi}\left(\frac{2+\omega+3\alpha\left(1+\omega\right)}{3+2\omega}\right)\left(\delta-\lambda\right) + \ddot{\lambda} + 2\left(1+\omega\right)\frac{\dot{\phi}}{\phi}\dot{\lambda}, \quad (5.21)$$

$$\ddot{\lambda} + \left(3H + 2\frac{\dot{\phi}}{\phi}\right)\dot{\lambda} + \left[\frac{n^2}{a^2} + \left(\frac{\ddot{\phi}}{\phi} + 3H\frac{\dot{\phi}}{\phi}\right)\right]\lambda = \frac{8\pi\left(1-3\alpha\right)\rho}{\left(3+2\omega\right)\phi}\delta + \frac{\dot{\phi}}{\phi}\frac{\dot{h}}{2}, \quad (5.22)$$

In addition, the conservation of energy-momentum tensor decompose in two equations, namely

$$2\dot{\delta} - \left(1+\alpha\right)\left(\dot{h} - 2U\right) = 0\ , \quad (5.23)$$

$$\left(1+\alpha\right)\left(\dot{U} + \left(2-3\alpha\right)HU\right) = \alpha\frac{n^2}{a^2}\delta. \quad (5.24)$$

where again n represents the number of the Fourier mode. In the long wavelength limit $n \to 0$, Eq. (5.24) shows that the perturbation of the four velocity decouples. For a stiff matter fluid, $\alpha = 1$, it becomes a growing mode with

$$U \propto a. \quad (5.25)$$

This growing mode has nothing to do with BD's extra scalar degree of freedom. Eqs. (5.23)-(5.24) come from perturbing the conservation of the energy-momentum Eq. (2.17), which is identical in GR. For equation of state lower than $\alpha < 2/3$, such as for radiation, we have a decaying mode and can ignore it by setting $U = 0$ that in turn implies $2\delta = \left(1+\alpha\right)h$. In contrast, in our case $\alpha = 1$ the growing mode together with Eq. (5.23) implies

$$\delta = h - \frac{4}{3}U_0 t^{3/2}, \quad (5.26)$$

with U_0 a constant of integration. For this reason, we will retain this inhomogeneous term.

Using the background expressions (Eq. (5.16)), the long wavelength limit ($n \to 0$), and Eq. (5.26), the dynamical system simplifies to

$$\frac{\ddot{h}}{2}+\frac{\dot{h}}{2t}+\frac{(5+4\omega)}{4t^2}h=\ddot{\lambda}-\frac{2\,(1+\omega)\,\dot{\lambda}}{t}+\frac{(5+4\omega)}{4t^2}\lambda+\frac{(5+4\omega)\,U_0}{3t^{1/2}}, \quad (5.27)$$

$$-\frac{\dot{h}}{2t}+\frac{h}{2t^2}=\ddot{\lambda}-\frac{\dot{\lambda}}{2t}+\frac{\lambda}{2t^2}+\frac{2U_0}{3t^{1/2}}. \quad (5.28)$$

These equations admit a solution under the form,

$$h=h_0t^m+f_0t^{\frac{3}{2}}, \quad \lambda=\lambda_0t^m+g_0t^{\frac{3}{2}}, \quad (5.29)$$

where λ_0, h_0, f_0 and g_0 are constants. Equating the power in the time parameter and the coefficients of the polynomials, we obtain a set of equations connecting these different constants of integration, namely

$$\left\{m^2+\frac{(5+4\omega)}{2}\right\}h_0=2\left\{m^2-(3+2\omega)m+\frac{(5+4\omega)}{4}\right\}\lambda_0, \quad (5.30)$$

$$(1-m)h_0=2\left\{m^2-\frac{3}{2}m+\frac{1}{2}\right\}\lambda_0, \quad (5.31)$$

$$f_0=-\frac{2}{3}(3+4\omega)g_0+\frac{8}{27}(5+4\omega)U_0, \quad (5.32)$$

$$f_0=-2g_0+\frac{8}{3}U_0. \quad (5.33)$$

The constants h_0 and λ_0 give the homogeneous modes, while f_0 and g_0 give the inhomogeneous modes associated with the growing mode U. The homogeneous mode admit four power solution given by $m=0,-1,\frac{1}{2}$ and 1. The solutions corresponding to $m=0,-1$ are connected with the residual gauge freedom typical of the synchronous gauge. A remarkable novelty is that the physical solutions correspond only to growing modes. In fact, these modes appear also in the long wavelength limit of the radiative cosmological model both in the GR and BD theories [89, 90]. However, there are two interesting aspects connected with these modes: there is no dependence on ω, and when

$m = 1$, $h_0 = \lambda_0 = 0$, while for $m = \frac{1}{2}$, $h = 0$ and λ_0 is arbitrary. Thus, the most important perturbative modes are the inhomogeneous modes, which are represented by $h = f_0 t^{3/2}$, $\lambda = g_0 t^{3/2}$, with

$$f_0 = \frac{2}{9}\frac{(7+2\omega)}{\omega}U_0, \tag{5.34}$$

$$g_0 = -\frac{4}{9}\frac{(7+6\omega)}{\omega}U_0. \tag{5.35}$$

An important feature of these inhomogeneous solutions is that the perturbation grows very quickly with the scale factor, $\delta \propto a^3$. It is also interesting to contrast with the same situation in GR where this inhomogeneous modes identically cancel for the pure stiff matter case [91, 92]. As a final remark, we should stress that these homogeneous and inhomogeneous modes have a well defined $\omega \to \infty$ limit. However, these solutions depend on the background solution, which is inconsistent in this limit.

5.3 Dynamical system analysis

The power law solution for stiff matter fluid (5.10) in BD has distinct features compared with Gurevich's families with $\alpha \neq 1$ displayed in Sec. 2.3. In order to compare these solutions, we perform a dynamical system analysis. It is convenient to define the Hubble factor $H = \frac{\dot{a}}{a}$ and its analogous for the scalar field, namely $F = \frac{\dot{\phi}}{\phi}$. Restricting ourselves to the stiff matter ($p = \rho$), we can combine the hamiltonian constraint Eq. (2.20) with Eq. (2.21) and Eq. (2.22), obtaining the following dynamical autonomous system:

$$\dot{H} = -\frac{6}{(3+2\omega)}\left((1+\omega)H^2 + \frac{\omega}{3}HF + \frac{\omega}{12}F^2\right), \tag{5.36}$$

$$\dot{F} = -\frac{6}{(3+2\omega)}\left(H^2 + \frac{(5+2\omega)}{2}HF + \frac{(3+\omega)}{6}F^2\right), \tag{5.37}$$

In fact, note that Eq.s (5.36)-(5.37) can simultaneously describe the stiff matter and the vacuum ($p = \rho = 0$). It is easy to check that there are two fixed points for this dynamical system

$$H = F = 0 \quad \text{corresponding to the Minkowski case,} \tag{5.38}$$

$$H = -\frac{F}{3} \quad \text{with} \quad \omega = -\frac{4}{3}. \tag{5.39}$$

We can also find the invariant rays defined by the condition $F = qH$ with q constant, which correspond to power law solutions of the system. Using Eq. (5.10), this condition translates into $q = \frac{s}{r}$, where r and s are the powers in time of the scale factor and of the scalar field, respectively. Imposing this condition and combining the resulting expressions we find the following third order polynomial for q

$$(q+2)\left(\frac{\omega}{2}q^2 - 3q - 3\right) = 0,$$

with solutions given by

$$q = -2, \quad q_{\pm} = \frac{3}{\omega}\left(1 \pm \sqrt{1 + \frac{2}{3}\omega}\right). \tag{5.40}$$

The first root corresponds to the power law solution found previously, for which gravity is attractive only if $\omega < -\frac{3}{2}$. Indeed, using $F = qH$, the constraint Eq. (2.20) reads

$$\left(-\frac{\omega}{2}q^2 + 3q + 3\right)H^2 = \frac{8\pi\rho}{\phi}. \tag{5.41}$$

One can immediately see that if $q = -2$ then the energy density is positive only for $2\omega + 3 < 0$ as already argued in Eq. (5.14). The other two roots correspond to the vacuum solution. Again, Eq. (5.41) shows that for $q = q_{\pm}$ the left hand-side of the above equation vanishes implying that $\rho = 0$. Note also that the invariant rays $q = q_{\pm}$ disappear when $\omega < -\frac{3}{2}$ (the roots become imaginary). Varying ω into negative values makes the two $q_{\pm}$ rays collapse into $q = -2$ when $\omega = -\frac{3}{2}$. For $\omega < -\frac{3}{2}$ only the $q = -2$ invariant ray remains (see Fig. 5.1).

The $q = -2$ invariant ray does not depend on ω which means that is insensitive to the $\omega \to \infty$ limit. On the other hand, the $q = q_{\pm}$ decays as

$$\lim_{\omega \to \infty} q_{\pm} = \pm\sqrt{\frac{6}{\omega}}. \tag{5.42}$$

Thus, since $F = qH$, for arbitrary finite values of H we have $\dot{\phi} \to 0$ in the $\omega \to \infty$ limit. Naively, one could expect that a vacuum solution with $\dot{\phi} \to 0$ should approach the Minkowski spacetime. However, the term $\omega\dot{\phi}^2$ does not go to zero in this limit producing a power law expansion with $a \propto t^{1/3}$. Indeed, it has been shown in Ref. [93] that in this regime the $\omega\dot{\phi}^2$ term behaves as an effective stiff matter like term, which is responsible for the $a \propto t^{1/3}$ evolution of the scale factor.

5.3.1 Singular Points at Infinity

The stability of the fixed point at the origin of the phase space can be inferred directly from the phase space diagrams. However, the stability of the invariant rays of the dynamical system must be analyzed at infinity. For this purpose we use the Poincaré central projection method [94]-[96], using the coordinate transformation

$$H = \frac{h}{z}, \quad F = \frac{f}{z}, \qquad \text{with } h^2 + f^2 + z^2 = 1. \tag{5.43}$$

Eqs. (5.36)–(5.37) can be recast as $P(H,F)dF - Q(H,F)dH = 0$, which combined with equation Eq. (5.43) gives us

$$-zQdh + zPdf + (hQ - fP)\,dz = 0. \tag{5.44}$$

where the functions $P(h,f,z)$ and $Q(h,f,z)$ are given by

$$P(h,f,z) = -\frac{6}{(3+2\omega)}\left((1+\omega)h^2 + \frac{\omega}{3}hf + \frac{\omega}{12}f^2\right), \tag{5.45}$$

$$Q(h,f,z) = -\frac{6}{(3+2\omega)}\left(h^2 + \frac{(5+2\omega)}{2}hf + \frac{(3+\omega)}{6}f^2\right). \tag{5.46}$$

Collecting all terms we can explicitly write Eq. (5.44) in terms of the projective coordinates as

$$\begin{aligned} &z\Big[6h^2 + 3(2\omega+5)hf + (3+\omega)f^2\Big]dh \\ &-z\left[6(1+\omega)h^2 + 2\omega hf + \frac{\omega}{2}f^2\right]df \\ &-\left[6h^3 - \frac{\omega}{2}f^3 + (3-\omega)hf^2 + 9h^2 f\right]dz = 0. \end{aligned} \tag{5.47}$$

The singular points at infinity have projective coordinates in the plane $(h, f, z = 0)$. Given equations (5.43) and (5.47), they are solutions of the system:

$$\begin{aligned} &h^2 + f^2 = 1, \\ &6h^3 - \frac{\omega}{2}f^3 + (3-\omega)hf^2 + 9h^2 f = 0. \end{aligned} \tag{5.48}$$

In order to find the invariant rays, we substitute $\frac{f}{h} = q$ in the system Eq.s (5.48). As expected, there are three invariant rays

$$q = -2: \qquad h = \pm\frac{1}{\sqrt{5}}, \qquad f = qh, \tag{5.49}$$

$$q_+ = \frac{3}{\omega}\left(1 + \sqrt{1 + \frac{2}{3}\omega}\right): \qquad h = \frac{1}{\sqrt{1+q_+^2}}, \qquad f = q_+ h, \tag{5.50}$$

$$q_- = \frac{3}{\omega}\left(1 - \sqrt{1 + \frac{2}{3}\omega}\right): \qquad h = \frac{1}{\sqrt{1+q_-^2}}, \qquad f = q_- h. \tag{5.51}$$

The analytical expressions of the solutions of the scale factor and the scalar field that correspond to these rays are given by

For $q = -2$:

$$a(t) \propto t^{1/2}, \quad \phi(t) \propto t^{-1}, \tag{5.52}$$

For $q_+ = \frac{3}{\omega}\left(1 + \sqrt{1 + \frac{2}{3}\omega}\right)$:

$$a(t) \propto t^{\omega(3+q_+)/3(4+3\omega)}, \quad \phi(t) \propto t^{(4-\omega q_+)/(4+3\omega)}, \tag{5.53}$$

For $q_- = \frac{3}{\omega}\left(1 - \sqrt{1 + \frac{2}{3}\omega}\right)$:

$$a(t) \propto t^{\omega(3+q_-)/3(4+3\omega)}, \quad \phi(t) \propto t^{(4-\omega q_-)/(4+3\omega)}. \tag{5.54}$$

The phase portrait for six different values of ω: -5, $-4/3$, -1, 1, 50 and 500 are plotted in Fig. 5.1. As mentioned before, the two spread invariant rays for $\omega > -3/2$ are related to the $q_\pm$ vacuum solution ($\rho = 0$), while the invariant ray in the middle is for $q = -2$, which corresponds to our solution Eq. (5.10) for $\omega < -3/2$. Increasing the value of ω makes the $q_\pm$ invariant rays to move away from the $q = -2$ invariant ray. As can be seen by Eq. (5.41), the region between the two invariant rays $q_\pm$ corresponds to negative values of the energy density, hence should be excluded on physical basis. Additionally, for large values of ω the $q_\pm$ invariant rays tend to lay along the $F = 0$ line, which represent the $\omega \to \infty$ limit.

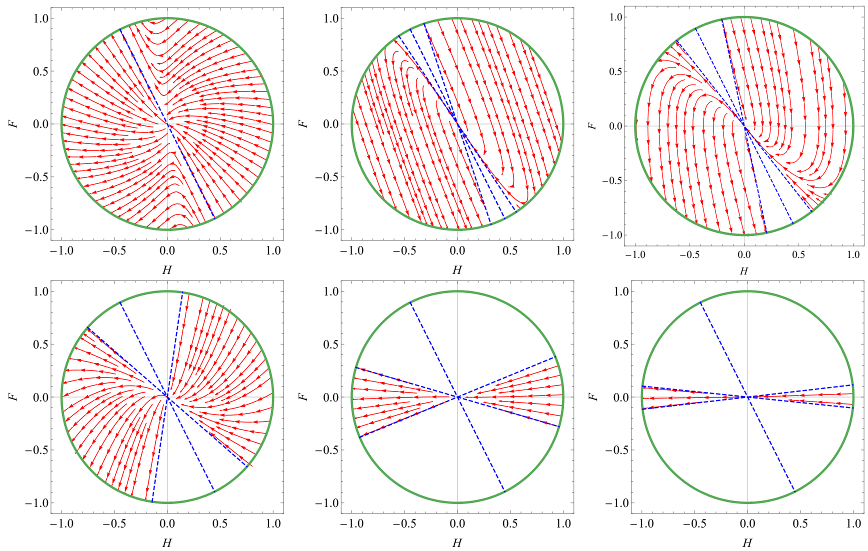

Figure 5.1: Compacted phase portraits using the variables h, f that are respectively related to H and F by Eq. (5.43). Each portrait uses a different value of BD parameter ω. In particular, we used the values $\omega = -5, -4/3, -1, 1, 50, 500$ respectively from the top left to the bottom right. The unity circle corresponds to the projected singular points at infinity. The dashed straight lines depict the invariant rays Eqs. (5.49-5.51) and the empty region has been excluded since it corresponds to the unphysical situation of negative values of the energy density.

Chapter 6

Cosmological singularities

In this section we provide a historical background and the concepts of cosmological singularities, in particular sudden future singularities.

How to define a cosmological singularity? Intuitively, one expects that divergences of any physical or geometrical quantity would be a characteristic feature of singularities. Vladimir Belinski in [97] defines a cosmological singularity as "a singularity that is both (i) a singularity in time, i.e., such that the singular three-dimensional manifold is spacelike; and (ii) a curvature singularity, i.e., such that the curvature invariants together with invariant characteristics of matter fields (like the energy density), if any, diverge on this manifold." One could equivalently call such singularities "spacelike singularities".

The first exactly solvable cosmological models of Einstein's theory revealed the presence of a very remarkable phenomenon: the Big Bang singularity. The initial singularity (along with the flatness and horizon problems mentioned in the Section 3) is one of the important problems of standard cosmology. Since the time the Big Bang singularity was discovered in 1922 by Alexander Friedmann [98], a fundamental question was formulated by Lev Landau in 1959: whether this phenomenon is due to the special simplifying assumptions underlying the exactly solvable models or whether a singularity is a general property of the Einstein equations?

This question was answered by V. Belinski, I. Khalatnikov and E. Lifshitz (BKL) [99, 100, 101] in 1969: a singularity is a general property of a generic cosmological solution of the classical gravitational equations and not a consequence of the special symmetric structure of the exact models. BKL were able to find the analytical structure of this generic solution and show that its behavior is of an extremely complex oscillatory character, of chaotic type. In other words, their work revealed the enigmatic phenomenon of an oscillatory approach to the singularity which has become known also as the Mixmaster Universe [102]. The model of the closed homogeneous but anisotropic universe with three degrees of freedom (Bianchi IX cosmological model) was used to demonstrate that the universe approaches the singularity in such a way that its contraction along two axes is accompanied by an expansion with respect to the third axis, and the axes change their roles according to a rather complicated law which reveals a chaotical behavior [100, 101, 103, 104].

Another way to look at singularities is *geodesic incompleteness*. A spacetime is considered singular if it is geodesically incomplete, i.e. there are freely-falling particles whose motion cannot be determined beyond a finite time, being after the point of reaching the singularity. In other words, the termination of such a geodesic is considered to be the singularity. For example, in the Big Bang cosmological model there is a casual singularity at $t = 0$, where all time-like geodesics have no extensions into the past.

The idea of geodesic incompleteness to describe singular spacetimes was introduce by R. Penrose in 1965 in his singularity theorem [105]. Moreover, he introduced the notion of closed trapped surface, a big conceptual contribution to the physics of the gravitational field. The Penrose theorem guarantees that some sort of geodesic incompleteness occurs inside any black hole whenever matter satisfies weak energy conditions: light rays are always focused together by gravity, never drawn apart, and this holds whenever the energy of matter is non-negative. Following Hawking's singularity theorem

[106] guarantees that the Big Bang has infinite density. This theorem is more restricted and only holds when matter obeys the dominant energy condition (the energy is larger than the pressure). Both theorems define a singularity to have geodesics that cannot be extended in a smooth manner.

We should make a clear distinction between a real physical singularity and fictitious ones. Fictitious singularities in time are unavoidably present in any synchronous system because the normal geodesics refocus, but all invariant characteristics of the gravitational field and matter are regular at such points. Singularities of this kind can be removed simply by passing to another system of coordinates. However if we cannot find a coordinate transformation that eliminates the singularity this doesn't mean the singularity is real - it might be we just haven't find the required transformation yet. In this case we need a quantity that independent of coordinate system. If this quantity becomes infinite then it must be infinite in all coordinates, and that means the singularity is real.

In the context of GR there are several such quantities, the simplest one is the Ricci scalar, but it is not very useful for vacuum solutions since the Ricci scalar is zero everywhere, as in the case of Schwarzschild and Kerr black holes. Rather useful example of such quantities is Kretschmann invariant

$$K = R_{ijkl}R^{ijkl}, \tag{6.1}$$

where R_{ijkl} is the Riemann curvature tensor. The existence of the singularity can be verified by noting if the Kretschmann scalar is infinite. This type of singularity is called *curvature singularity.*

For a general FLRW spacetime with metric (2.19) the Kretschmann scalar takes form

$$K = \frac{12\left(a\left(t\right)^2 a''\left(t\right)^2 + \left(k + a'\left(t\right)^2\right)^2\right)}{a\left(t\right)^4}. \tag{6.2}$$

For example, The Kretschmann scalar for a Schwarzschild black hole is given by

$$K = \frac{48G^2M^2}{c^4r^6},$$

where where G is the gravitational constant. At $r = 0$ the curvature becomes infinite, indicating the presence of a singularity. For a long time this solution was considered as non-physical. However, a greater understanding of GR led to the realization that such singularities were a generic feature of the theory and not just an exotic special case.

Until the end of 1990s almost all the discussions about classical and quantum cosmology of singularities were devoted to the Big Bang and Big Crunch singularities, which are characterized by a vanishing cosmological radius. The situation was changed after the discovery of the phenomenon of the cosmic acceleration. This discovery was the starting point for the formulation of cosmological models containing dark energy, which, for its specific properties, was considered responsible for the accelerated expansion of the Universe. The fundamental feature of the dark energy is that it possesses a pressure p such that the strong energy condition $\rho + 3p > 0$ is violated (here ρ is the energy density). The construction of different cosmological models, describing dark energy, has attracted the attention of researchers to the fact that other types of cosmological singularities do exist. First of all, one should mention the Big Rip singularity [107, 108] arising in the models where the phantom dark energy [109] is present. Under phantom dark energy one understands the substance whose pressure is negative and has an absolute value bigger than its energy density. Such a singularity is characterized by infinite values of the cosmological radius (scale factor), of its time derivative, of the Hubble parameter and its time derivative and, hence, of its energy density and pressure.

6.1 Sudden (soft) future singularities

Kinematical investigations of Friedmann cosmologies have raised the question about the possibility of existence of sudden future singularity occurrence [110], characterized by a diverging $\ddot{a}$ whereas both the scale factor a and $\dot{a}$ are finite. Then the Hubble parameter $H = \frac{\dot{a}}{a}$ and the energy density ρ are also finite, while the first derivative of the Hubble parameter and the pressure p diverge.

In the investigations devoted to sudden singularities one can distinguish three main topics. The first of them deals with the question of the compatibility of the models possessing soft singularities with observational data [111, 112, 113, 114]. The second direction is connected with the study of quantum effects [115]-[124]. Here one can see two subdirections: the study of quantum corrections to the effective Friedmann equation, which can eliminate classical singularities or, at least, change their form [117, 119, 125], and the study of solutions of the Wheeler-DeWitt equation for the quantum state of the universe in the presence of sudden singularities [119, 120, 121, 123] The third direction is connected with the opportunity of the crossing of sudden singularities in classical cosmology [122, 126, 127].

A particular feature of the sudden future singularities is their softness [126]. As the Christoffel symbols depend only on the first derivative of the scale factor, they are regular at these singularities. Hence, the geodesics have a good behavior and they can cross the singularity [126]. One can argue that the particles crossing the singularity will generate the geometry of the spacetime, providing in such a way a soft rebirth of the universe after the singularity crossing [127]. Note that the opportunity of crossing of some kind of cosmological singularities were noticed already in the early paper by Tipler [128]. A close idea of integrable singularities in black holes, which can give origin to a cosmogenesis, was recently put forward in [129]-[130].

Another remarkable feature of the soft future singularities is their capacity to induce changes in the equations of state of the matter present in a universe under consideration. Moreover, the form of the matter Lagrangian can also be changed. These effects were considered in [131, 133, 132]. The effects of the matter transformation occur sometimes also without singularities, but only in the presence of some non-analyticities in the geometry of the spacetime. These phenomena have also some kinship with those of the singularity crossing [133, 132]. In the section 8 we shall study another aspect of the presence of soft singularities and non-analyticities of geometry - we are interested in the behavior of quantum particles in the vicinity of these particular spacetime points.

6.2 Classification of the future cosmological singularities

In this section we present a classification of future cosmological singularities, following [133] and [134]. We consider the dark energy universe models which contain finite-time future singularities.

The *type I* singularity is so called Big Rip singularity, characterized by $a \to \infty$, $\dot{a} \to \infty$, $H \to \infty$, $\rho \to \infty$ and $|p| \to \infty$ at some finite moment of time $t \to t_{BR}$. As was mention before, this type of singularity arises in the models where the phantom dark energy is present.

The *type II* singularity corresponds to the sudden singularities, which arises at some finite moment of time $t \to t_S$, when $a \to a_S$, $\dot{a} \to \dot{a}_S$, $H \to H_S$ and $\rho \to \rho_S$, but the acceleration of the universe $\ddot{a}$ and the first time derivative of the Hubble parameter $\dot{H}$ diverge to minus infinity and the pressure tends to plus infinity $p \to \infty$. A particular case of sudden future singularity is the Big Brake singularity. At this singularity, the time derivative of the cosmological radius, the Hubble variable and the energy density are equal exactly to zero. As $\ddot{a} \to -\infty$, this works as an infinitely strong "brake," forcing the derivative of the scale factor to go to zero. The evolution of the scale factor stops. It can arise in tachyonic cosmological models [135] with a particular potential, it was also noticed that the big-brake singularity can arise in more simple cosmological models, such as a universe filled with a perfect fluid obeying the equation of state $p = \frac{A}{\rho}$, where A is a positive constant (anti-Chaplygin gas).

The *type III* singularity appears in the model with equation of state $p = -\rho - A\rho^{\alpha}$ and sometimes is called "finite scale factor singularity". The name speaks for itself - this type of singularity has a finite scale factor $a \to a_S$, but this singularity is different from the sudden future singularity in the sense that the rest of the cosmological parameters diverge, that is $\dot{a} \to \infty$, $\rho \to \infty$, and $|p| \to \infty$.

The *type IV* is a very soft singularity at which a is finite, ρ and p tend to zero, but higher derivatives of the Hubble parameter diverge. These singularities sometimes are called Big Separation singularities.

The *type V* is a w-singularities, characterized by finite scale factor a, vanishing energy density ρ and pressure p, and regular higher time derivatives of the Hubble parameter. The only singular behavior appears in a time-dependent barotropic index $w(t)$. In the context of physical theories w-singularities may appear in $f(R)$ gravity [136], in scalar field models [137], and in brane cosmologies [138]. In paper [139] was found an interesting duality between the big-bang and the w-singularity which refers to the pressure, the energy density and the barotropic w-index.

The singularities which stay apart from this classification are the traditional Big Bang and Big Crunch singularities, sometimes called *type 0* singularities [140].

In the next sections we describe the notion of particle at approaching the Big Bang - Big Crunch, Big Rip and Big Brake singularities and others nonanalyticities in the geometry of the spacetime.

Chapter 7

Particles and cosmological singularities

It is well known that the very notion of particle becomes complicated when one considers the quantum field theory on a curved spacetime background [142, 143, 144]. Let us recapitulate the general procedure for the definition of the particles on the example of a scalar field filling a flat Friedmann universe with the metric

$$ds^2 = dt^2 - a^2(t)dl^2. \tag{7.1}$$

The Klein-Gordon equation for the minimally coupled scalar field ϕ with the potential $V(\phi)$ is

$$\Box\phi + V'(\phi) = 0, \tag{7.2}$$

where $\Box$ is the d'Alambertian. One can consider a spatially homogeneous solution of this equation ϕ_0, depending only on time t as a classical background. A small deviation from this background solution can be represented as a sum of Fourier harmonics satisfying linearized equations

$$\ddot{\phi}(\vec{k},t) + 3\frac{\dot{a}}{a}\dot{\phi}(\vec{k},t) + \frac{\vec{k}^2}{a^2}\phi(\vec{k},(t)) + V''(\phi_0(t))\phi(\vec{k},(t)) = 0. \tag{7.3}$$

The corresponding quantized field is represented in the following form:

$$\hat{\phi}(\vec{x},t) = \int d^3\vec{k}(\hat{a}(\vec{k})u(k,t)e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+(\vec{k})u^*(k,t)e^{-i\vec{k}\cdot\vec{x}}), \tag{7.4}$$

where the creation and the annihilation operators satisfy the standard commutation relations:

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = \delta(\vec{k} - \vec{k}'), \tag{7.5}$$

while the basis functions u satisfy the linearized equation (7.3). These basis functions should be normalized so that the canonical commutation relations between the field ϕ and its canonically conjugate momentum $\hat{\mathcal{P}}$ were satisfied

$$[\hat{\phi}(\vec{x},t), \hat{\mathcal{P}}(\vec{y},t')] = i\delta(\vec{x} - \vec{y}). \tag{7.6}$$

Taking into account the fact that for the minimally coupled scalar field the momentum is

$$\hat{\mathcal{P}}(\vec{x},t) = a^3\dot{\hat{\phi}}(\vec{x},t) \tag{7.7}$$

the commutation relation (7.5) and the Fourier representation for the Dirac delta function, one easily shows that the relation (7.6) is satisfied if

$$u(k,t)\dot{u}^*(k,t) - u^*(k,t)\dot{u}(k,t) = \frac{i}{(2\pi)^3 a^3(t)}. \tag{7.8}$$

The linearized equation (7.3) has two independent solutions. As for functions u, one can take different linear combinations of these solutions chosen in such a manner that the Wronskian relation (7.8) is satisfied. Different choices of these functions determine different choices of the creation and the annihilation operators and different vacuum states on which the Fock spaces can be constructed. In the Minkowski spacetime a preferable choice simply corresponds to the plane waves. In the de Sitter spacetime it is common to define the Bunch-Davies vacuum [145], which in the limit of large wave numbers is close to the Minkowski vacuum. In any case, in order to have some definition of particle it is necessary to obtain two independent non-singular solutions of Eq. (7.3). However, it is a non-trivial requirement in the situations when a singularity or other kind of irregularity of the spacetime geometry occurs. One can easily understand that this is connected with the presence of the time-dependent scale factor $a(t)$ in the right-hand side of the relation (7.8).

It is convenient also to construct explicitly the vacuum state for quantum particles as a Gaussian function of the corresponding variable. Let us introduce an operator

$$\hat{f}(\vec{k},t) = (2\pi)^3(\hat{a}(\vec{k})u(k,t) + \hat{a}^+(-\vec{k})u^*(k,t)). \tag{7.9}$$

Its canonically conjugate momentum is

$$\hat{p}(\vec{k},t) = a^3(t)(2\pi)^3(\hat{a}(\vec{k})\dot{u}(k,t) + \hat{a}^+(-\vec{k})\dot{u}^*(k,t)). \tag{7.10}$$

Now we can express the annihilation operator as

$$\hat{a}(\vec{k}) = i\hat{p}(\vec{k},t)u^*(k,t) - ia^3(t)\hat{f}(\vec{k},t)\dot{u}^*(k,t), \tag{7.11}$$

where we have used the Wronskian relation (7.8). Representing the operators $\hat{f}$ and $\hat{p}$ as

$$\hat{f} \to f, \quad \hat{p} \to -i\frac{d}{df}, \tag{7.12}$$

one can write down the equation for the corresponding vacuum state in the following form:

$$\left(u^*\frac{d}{df} - ia^3\dot{u}^*f\right)\Psi_0(f) = 0. \tag{7.13}$$

The normalized solution to Eq. (7.13) is (up to a non-essential constant)

$$\Psi_0(f) = \frac{1}{\sqrt{|u(k,t)|}}\exp\left(\frac{ia^3(t)\dot{u}^*(k,t)f^2}{2u^*(k,t)}\right). \tag{7.14}$$

7.1 Big Bang – Big Crunch, Big Rip and particles

At the Big Bang or the Big Crunch singularity a universe has a vanishing volume or in the case of homogeneous and isotropic Friedmann universe, which we consider in this section, the vanishing scale factor a. This means that the Wronskian, which is inversely proportional to a^3 (see Eq. (7.8)), becomes singular. This points out that it could be impossible to construct the non-singular basis functions in the vicinity of the singularity, and, correspondingly, one cannot introduce a Fock vacuum and the operators of creation and annihilation. To confirm this statement let us consider a simple case of a flat Friedmann universe filled with a perfect fluid with the equation of state

$$p = w\rho, \tag{7.15}$$

where p is the pressure, ρ is the energy density and w is a constant such that $-\frac{1}{3} < w \leq 1$. (Note that in the first part of the thesis the parameter w was denoted as α in order to distinguish it from the Brans-Dicke parameter ω). The law of expansion of the universe is

$$a(t) = a_0 t^{\frac{2}{3(1+w)}}. \tag{7.16}$$

We can consider, for example, a free massive scalar field living in this universe. Then Eq. (7.3) looks as

$$\ddot{u}(\vec{k},t) + \frac{2}{(1+w)t}\dot{u}(\vec{k},t) + \frac{k^2}{a_0^2 t^{\frac{4}{3(1+w)}}}u(\vec{k},t) + m^2 u(\vec{k},t) = 0. \tag{7.17}$$

Obviously, considering Eq. (7.17) at $t \to 0$, we can neglect the massive term with respect to the term inversely proportional to $t^{\frac{4}{3(1+w)}}$. After this it is easy to find that

$$u(\vec{k},t) = c_1 t^{\frac{w-1}{2(1+w)}} J_{\frac{1-w}{2(1+w)}} \left(\frac{3k(1+w)}{a_0(1+3w)} t^{\frac{(1-w)(1+3w)}{(1+3w)^2}} \right)$$
$$+ c_2 t^{\frac{w-1}{2(1+w)}} Y_{\frac{1-w}{2(1+w)}} \left(\frac{3k(1+w)}{a_0(1+3w)} t^{\frac{(1-w)(1+3w)}{(1+3w)^2}} \right). \quad (7.18)$$

Here, J and Y are the corresponding Bessel functions. We see that the term, proportional to the function Y becomes singular when $t \to 0$ and, hence, we do not have two independent non-singular solutions for the basis functions and cannot construct the vacuum and the Fock space. Note, that this conclusion is valid even if for the model under consideration one manages to describe the Big Bang - Big Crunch singularity crossing, using some of the approaches mentioned in the Introduction.

Now, let us consider an extreme opposite case - the Big Rip singularity [107, 108, 109]. The simplest model, where this singularity arises, is the Friedmann universe filled with a perfect fluid with a constant equation of state parameter w such that $w < -1$. In this case the scale factor behaves as

$$a(t) = a_0(-t)^{\frac{2}{3(1+w)}}, \quad (7.19)$$

and when $t \to 0_-$ the scale factor tends to ∞. The equation for the perturbations of the massive scalar field on this background have the same form as Eq. (7.17), but now we can neglect the term $\frac{k^2}{a_0^2 t^{\frac{4}{3(1+w)}}} u(\vec{k},t)$, which tends to zero as $t \to 0_-$. Thus, the solution of the corresponding equation is

$$u(\vec{k},t) = c_1(-t)^{\frac{w-1}{2(1+w)}} J_{\frac{w-1}{2(1+w)}}(-mt)$$
$$+ c_2(-t)^{\frac{w-1}{2(1+w)}} Y_{\frac{w-1}{2(1+w)}}(-mt). \quad (7.20)$$

Both independent solutions are now regular at $t \to 0_-$ and we can construct the Fock vacuum. Thus, nothing special happens with particles when universe enters into the Big Rip singularity. Let us construct this vacuum state in the vicinity of the singularity explicitly, using the formula (7.14). In the vicinity of the Big Rip we can write down the basis function using the independent solutions (7.20) and keeping only the leading terms as follows:

$$u(\vec{k},t) = A + iB(-t)^{\frac{w-1}{1+w}}. \tag{7.21}$$

This function should satisfy the Wronskian relation (7.8), with the scale factor given by the formula (7.19). It means that the constants A and B satisfy the equation

$$AB = \frac{(1+w)}{(2\pi)^3 a_0^3 (w-1)}. \tag{7.22}$$

Then,

$$\Psi_0(f) \sim \frac{1}{A} \exp\left(-\frac{1}{(2\pi)^3 A^2} f^2\right). \tag{7.23}$$

The Gaussian exponent is well defined in the vicinity of the Big Rip singularity. We still have the freedom to choose the value of the positive constant A. We know, for example, that in the case of the de Sitter spacetime, one can fix an analogous freedom by requiring that the vacuum has a standard Minkowski form in the infinitely remote past. Here, we cannot follow the evolution of our basis function to the past infinity and, thus, we leave the value of the constant A unspecified. However, for any choice of this constant, the function (7.23) has a regular behavior. Let us note that at least up to our knowledge there are no attempts to describe the Big Rip singularity crossing. Thus, the regular behavior of the quantum particles approaching the Big Rip singularity does not mean that such a singularity can be crossed, or they can survive such a crossing. Nevertheless, the fact of the regular behavior of functions, entering into the formulas (7.20) and (7.23) looks interesting.

Let us consider a slightly more complicated situation when the evolution of type (7.19) is provided by the presence of the phantom scalar field with the negative kinetic term and an exponential potential:

$$L = -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V_0 \exp(-\alpha\phi). \tag{7.24}$$

The Friedmann equation is now

$$\frac{\dot{a}^2}{a^2} = -\frac{1}{2}\dot{\phi}^2 + V_0 \exp(-\alpha\phi), \tag{7.25}$$

while the Klein-Gordon equation is

$$\Box\phi + \alpha V_0 \exp(-\alpha\phi) = 0. \tag{7.26}$$

If we choose

$$V_0 = \frac{2(1-w)}{9(1+w)^2} \tag{7.27}$$

and

$$\alpha = 3\sqrt{-(1+w)}, \tag{7.28}$$

then we have the evolution (7.19) and the background solution for the phantom scalar field is

$$\phi(t) = \frac{2}{3\sqrt{-(1+w)}} \ln(-t). \tag{7.29}$$

Before writing down the equation for the linear perturbations we should substitute into the Klein-Gordon equation (7.26) the expression for $\frac{\dot{a}}{a}$ following from the Friedmann equation (7.25). Then we have the equation which includes only the scalar field and its derivatives. The equation for the linear perturbations is now

$$\ddot{u}(\vec{k},t) + \frac{1-w}{(1+w)t}\dot{u}(\vec{k},t) + \frac{k^2}{a_0^2 t^{\frac{4}{3(1+w)}}} u(\vec{k},t) + \frac{1-w}{(1+w)t^2} u(\vec{k},t) = 0. \tag{7.30}$$

In the vicinity of the Big Rip singularity $t \to 0_-$, the solution of Eq. (7.30) behaves as

$$u(\vec{k}, t) = c_1(-t)^{\kappa_1} + c_2(-t)^{\kappa_2}, \tag{7.31}$$

where

$$\kappa_1 = \frac{w}{1+w} + \sqrt{\frac{2w^2-1}{(1+w)^2}} > 0, \tag{7.32}$$

$$\kappa_2 = \frac{w}{1+w} - \sqrt{\frac{2w^2-1}{(1+w)^2}} < 0. \tag{7.33}$$

Thus, the second solution in (7.31) is singular as $t \to 0_-$ and we cannot construct the Fock space for it.

Chapter 8

Born-Infeld-like fields and particles

In this section we consider different scenarios of the evolution of the Universe, where the singularities or some nonanalyticities in the geometry of the spacetime are present, trying to answer the following question: is it possible to conserve some kind of notion of particle corresponding to a chosen quantum field present in the universe when the latter approaches the singularity? We study scalar fields with different types of Lagrangians, writing down the second-order differential equations for the linear perturbations of these fields in the vicinity of a singularity. If both independent solutions are regular, we construct the vacuum state for quantum particles as a Gaussian function of the corresponding variable. If at least one of two independent solutions has a singular asymptotic behavior, then we cannot define the creation and the annihilation operators and construct the vacuum. This means that the very notion of particle loses sense. In the case of the model of the universe described by the tachyon field with a special trigonometric potential, where the big brake singularity occurs, we see that the (pseudo) tachyon particles do not pass through this singularity. Adding to this model some quantity of dust, we slightly change the characteristics of this singularity and tachyon particles survive.

8.1 Tachyon scalar field cosmology

The discovery of cosmic acceleration [7] stimulated searches of the so-called dark energy responsible for this effect [8, 9]. The simplest candidate for the dark energy is a positive time-independent cosmological constant Λ with $w = -1$. However, it is difficult to understand why the cosmological constant is about 120 orders of magnitude smaller than its natural expectation, i.e., the Planck energy density (so-called *cosmological constant problem*).

One of the possible candidates for this role was tachyon field, arising in string theories [135, 146, 147, 148]. As a matter of fact what is called tachyon field is a modification of an old idea of Born and Infeld [149], that the kinetic term of a field can have a non-polynomial form. The Lagrangian of the tachyon field T has the form

$$L = -V(T)\sqrt{1 - g^{\mu\nu}T_{,\mu}T_{,\nu}}, \tag{8.1}$$

which for a spatially homogeneous field becomes

$$L = -V(T)\sqrt{1 - \dot{T}^2}. \tag{8.2}$$

The energy density corresponding to (8.2) is

$$\rho = \frac{V(T)}{\sqrt{1 - \dot{T}^2}}, \tag{8.3}$$

while the pressure is negative and equal to

$$p = -V(T)\sqrt{1 - \dot{T}^2}. \tag{8.4}$$

The negativity of the pressure makes the tachyon field a good candidate for the dark energy role. The field equation for the tachyon field is

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_{,T}}{V(T)} = 0, \tag{8.5}$$

where $H(t) = \frac{\dot{a}(t)}{a(t)}$.

There is also a great freedom for the choice of the potential $V(T)$. One can try and find a potential $V(T)$ so that, for certain suitably chosen initial conditions on the tachyon field, the scale factor of the universe is precisely the given $a(t)$.

8.2 Tachyon models and soft singularities

In the paper [115] a very particular potential, depending on the trigonometrical functions was chosen:

$$V(T) = \frac{\Lambda\sqrt{1-(1+w)\cos^2\left(\frac{3}{2}\sqrt{\Lambda(1+w)}\right)T}}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda(1+w)}T\right]}, \tag{8.6}$$

where Λ is a positive constant and $-1 < w \leq 1$. What is the origin of this potential? If one consider a flat Friedmann model filled with the cosmological constant Λ and a perfect fluid with a constant barotropic index w then one can find an exact solution for the cosmological evolution. Then it is possible to reconstruct the potential $V(T)$ of the tachyon field generating this exact solution as a particular solution of the system which includes the Friedmann equation and Eq. (8.5). This potential is nothing but the potential (8.6) from the paper [115]. However, the dynamics of the Friedmann model based on the tachyon field with the potential (8.6) is more rich than that of the model with two fluids, because the model with tachyon has more degrees of freedom. The case when the parameter w is positive is particularly interesting. To study this case it is convenient to rewrite the Klein-Gordon-type equation (8.5) as a dynamical system of two first-order differential equations:

$$\dot{T} = s, \tag{8.7}$$

$$\dot{s} = -3\sqrt{V}\left(1 - s^2\right)^{\frac{3}{4}} s - \left(1 - s^2\right)\frac{V_{,T}}{V}. \tag{8.8}$$

The phase portrait for this dynamical system is presented on the Fig. 8.1, which was taken from the paper [115]. White regions in the phase diagram, where the Lagrangian and other quantities would become imaginary, are forbidden. One can see that the potential (8.6) is well defined inside the rectangle, where $-1 \leq s \leq 1$ and $T_3 \leq T \leq T_4$, with

$$T_3 = \frac{2}{3\sqrt{(1+w)\Lambda}} \arccos\frac{1}{\sqrt{1+w}}, \tag{8.9}$$

$$T_4 = \frac{2}{3\sqrt{(1+w)\Lambda}}\left(\pi - \arccos\frac{1}{\sqrt{1+w}}\right). \tag{8.10}$$

The analysis of this dynamical system shows that there are two families of the trajectories, one of them tends to the center of the rectangle, where $s = 0$ and $T = \frac{\pi}{3\sqrt{(\Lambda(1+w)}}$. Such a cosmological evolution is very close to one in the standard ΛCDM model. Another family includes the trajectories which tend to corners of our rectangle: one with $s = -1$ and $T = T_3$ and the symmetric one with $s = 1$ and $T = T_4$.

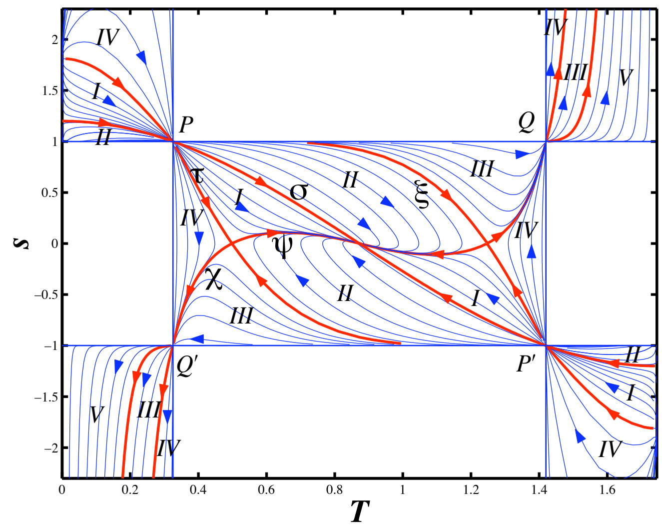

Figure 8.1: Phase portrait for the dynamical system of the model with a tachyon field and trigonometric potential for a positive w.

What happens with the universe approaching, for example, the lower left corner? The expression under the square root in the potential (8.6) tends to zero and the kinetic expression $\sqrt{1-s^2}$ tends to zero and it looks like we cannot cross the corner. At the same time it is easy to see there is no cosmological singularity here. Moreover, the differential equations are also regular. In paper [115] the only possible way out was suggested. The Lagrangian changes its form in such a way that the equations of motion conserve their form. The new Lagrangian is

$$L = W(T)\sqrt{\dot{T}^2 - 1}, \tag{8.11}$$

where

$$W(T) = \frac{\Lambda\sqrt{(1+w)\cos^2\left[\frac{3}{2}\sqrt{\Lambda\,(1+w)}T\right] - 1}}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda(1+w)}T\right]}, \tag{8.12}$$

and the new field (or a new form of the old field) is called pseudotachyon

[115]. This field arises when the universe enters into the left lower infinite strip on the figure. Note that the Friedmann equation for the universe filled with the pseudotachyon field is

$$\frac{\dot{a}^2}{a^2} = \frac{W(T)}{\sqrt{\dot{T}^2 - 1}}. \tag{8.13}$$

Let us describe in detail what happens with the field when it crosses the corner. The spatially homogeneous part of the field T behaves as

$$T = T_3 + \bar{T}, \tag{8.14}$$

where $\bar{T}$ is a small function, while

$$s = -1 + \bar{s}. \tag{8.15}$$

Substituting the formulas (8.14) and (8.15) into Eq. (8.5), we find that the functions $\bar{T}$ and $\bar{s}$ satisfy a simple equation

$$\frac{d\bar{s}}{d\bar{T}} = \frac{\bar{s}}{\bar{T}}. \tag{8.16}$$

Its general solution is

$$\bar{s} = C\bar{T}, \tag{8.17}$$

where C is a positive constant. Remembering that $s = \dot{T}$ and choosing (for convenience) that the moment of crossing is equal to $t = 0$ we can also note that our field crosses the corner so that

$$\bar{T} = -t, \tag{8.18}$$

and

$$\bar{s} = -Ct. \tag{8.19}$$

It is interesting to notice that in paper [114] the predictions of the model, suggested in paper [115], were compared with the supernovae type Ia data and it was discovered that there were cosmological trajectories going toward the corners which were compatible with these data.

Now, before going inside the strip to study the cosmological evolution there, let us consider what happens with particles during the transformation of the tachyon into the pseudotachyon. To do this, we add to Eq. (8.5) the terms responsible for the contribution of the spatial derivatives

$$\frac{\ddot{T}\left(1+\frac{1}{a^2}T_{,i}T_{,i}\right)}{1-\dot{T}^2+\frac{1}{a^2}T_{,i}T_{,i}}+3\frac{\dot{a}}{a}\dot{T}+\frac{V_{,T}}{V}$$
$$+\frac{\dot{a}a\dot{T}T_{,i}T_{,i}-2a^2\dot{T}\dot{T}_{,i}T_{,i}+T_{,i}T_{,j}T_{,ij}}{a^4\left(1-\dot{T}^2+\frac{1}{a^2}T_{,i}T_{,i}\right)}$$
$$-\frac{1}{a^2}\Delta T=0. \tag{8.20}$$

Now, expressing $\frac{\dot{a}}{a}$ through the Friedmann equation

$$\frac{\dot{a}^2}{a^2}=\rho,$$

and representing the tachyon field as

$$T=T_0+\tilde{T},$$

where T_0 is the solution of the tachyon field equation for the spatially homogeneous background mode and $\tilde{T}$ is the linear perturbation, we obtain the following equation for the linear perturbations

$$\frac{\ddot{\tilde{T}}}{1-\dot{T}_0^2} + \left(\frac{2\ddot{T}_0\dot{T}_0}{(1-\dot{T}_0^2)^2} + \frac{3\sqrt{V}(2-\dot{T}_0^2)}{2(1-\dot{T}_0^2)^{5/4}}\right)\dot{\tilde{T}} + \left(\frac{3V_{,T}\dot{T}_0}{2\sqrt{V}(1-\dot{T}_0^2)^{1/4}} + \frac{V_{,TT}}{V} - \frac{V_{,T}^2}{V^2} + \frac{k^2}{a^2}\right)\tilde{T} = 0. \tag{8.21}$$

Then we substitute the expressions (8.14) and (8.15) into Eq. (8.21) instead of T_0, and omitting subleading terms we obtain the following differential equation for the linear perturbations

$$\ddot{\tilde{T}} - \frac{1}{t}\dot{\tilde{T}} + \frac{C}{t}\tilde{T} = 0. \tag{8.22}$$

The solution is

$$\tilde{T} = c_1 t J_2\left(\sqrt{Ct}\right) + c_2 t Y_2\left(\sqrt{Ct}\right), \tag{8.23}$$

where J and Y are the Bessel functions. Both solutions are regular at $t \to 0$ and the particles should pass through the corner. The same analysis can be carried out in the upper left corner, where the pseudotachyon field is transformed into the tachyon field while the universe is expanding.

However, for the perturbations of the tachyon field the relations between the amplitudes of the models and their conjugate momenta differ from that for the minimally coupled scalar field (7.7). Indeed, due to the nonlinearity of the Lagrangian (8.1), this relation looks now

$$\mathcal{P}_{\tilde{T}} = \frac{V(T_0)}{\sqrt{1-\dot{T}_0{}^2}} a^3 \dot{\tilde{T}}. \tag{8.24}$$

This means that in the Wronskian relation instead of a^3 one has

$$a^3 \to \frac{V(T_0)}{\sqrt{1-\dot{T_0}^2}} a^3 \dot{\tilde{T}}. \tag{8.25}$$

Taking into account the Friedmann equation, we have

$$\mathcal{P}_{\tilde{T}} = \dot{a}^2 a \dot{\tilde{T}}. \tag{8.26}$$

Correspondingly the quantum state of the vacuum is represented by the function

$$\Psi_0(\tilde{T}) \sim \frac{1}{\sqrt{|u|}} \exp\left(i\dot{a}^2 a \frac{\dot{u}^*}{u*} \tilde{T}^2 \right). \tag{8.27}$$

Here, the factor $\dot{a}^2 a$ is a finite number at the crossing of the corner. As follows from the formula (8.23) in the vicinity of the corner the basis functions behave as

$$u = A + iBt^2, \tag{8.28}$$

where A and B are some constant, satisfying the normalization relation. We obtain that

$$\Psi_0(\tilde{T}) \sim \exp\left(-C(-t)\tilde{T}^2 \right), \tag{8.29}$$

where C is a positive constant. Thus, we see that there is a difference between this formula and the formula (7.23), obtained in the preceding section. Indeed, here the coefficient in front of $\tilde{T}^2$ is not a constant as it was in (7.23)) but is proportional to $-t$. It means that at the moment of the corner crossing the Gaussian function has the infinite dispersion. Then after the crossing at $t > 0$ it will have a form

$$\Psi_0(\tilde{T}) \sim \exp\left(-Ct\tilde{T}^2 \right). \tag{8.30}$$

Thus, in this case we have a regular basis functions in the vicinity of the corner, but at the passing through it the vacuum state in some manner disappear (one can interpret the infinite dispersion in this way), but immediately after the crossing we have a Fock space again. Perhaps, this momentary disappearance of the vacuum corresponds to the transformation of the particles of the tachyon field into the particles of the pseudotachyon field.

Let us remember what happens with the pseudotachyon field and the universe after the crossing the left lower corner. As it was described in [115] at some finite moment of time and at some finite value of the tachyon field the universe encounters the Big Brake singularity, where the scale factor has a finite value too, its time derivative is equal to zero, while the deceleration tends to infinity. Choosing the moment of arriving to the Big Brake as $t = 0$ we can write down the expressions for the pseudotachyon field and the cosmological scale factor as follows [127]:

$$T_0(t) = T_{BB} + \left(\frac{4}{3W(T_{BB})}\right)^{1/3} (-t)^{1/3}, \tag{8.31}$$

$$a(t) = a_{BB} - \frac{3}{4} a_{BB} \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3} (-t)^{4/3}. \tag{8.32}$$

Taking into account the fact that the Friedmann equation is given now by (8.13), the equation for the linear perturbation becomes slightly different from Eq. (8.21):

$$\begin{aligned} &\frac{\ddot{\tilde{T}}}{1-\dot{T}_0^2} + \left(\frac{2\ddot{T}_0\dot{T}_0}{(1-\dot{T}_0^2)^2} + \frac{3}{2}\frac{\sqrt{W}(2-\dot{T}_0^2)}{(\dot{T}_0^2-1)^{5/4}}\right)\dot{\tilde{T}} \\ &+ \left(\frac{3}{2}\frac{W_{,T}\dot{T}_0}{\sqrt{W}(\dot{T}_0^2-1)^{1/4}} + \frac{W_{,TT}}{W} - \frac{W_{,T}^2}{W^2} + \frac{k^2}{a_{BB}^2}\right)\tilde{T} \\ &= 0. \end{aligned} \tag{8.33}$$

Using the expression (8.31), we reduce Eq. (8.33) to the following simple form (keeping only the leading terms)

$$\ddot{\tilde{T}} + \frac{5}{3t}\dot{\tilde{T}} + \frac{B^2}{t^{\frac{5}{3}}}\tilde{T} = 0, \tag{8.34}$$

where

$$B^2 = -\frac{W_{,T}(T_{BB})}{16}\left(\frac{4}{3W(T_{BB})}\right)^{4/3} > 0. \tag{8.35}$$

The general solution of Eq. (8.34) is

$$\tilde{T}(t) = c_1 t^{-\frac{1}{3}} J_2\left(Bt^{\frac{1}{6}}\right) + c_2 t^{-\frac{1}{3}} Y_2\left(Bt^{\frac{1}{6}}\right). \tag{8.36}$$

Obviously, the second term in the right-hand side of Eq. (8.36) is singular at $t \to 0_-$ and we cannot use two independent solutions of the differential equation (8.34) to construct the Fock space. Thus, when approaching the Big Brake singularity the particles in some way disappear.

It is interesting to consider a little bit different situation when the universe encounters a more general soft singularity [131]. Suppose that our universe is filled not only with the tachyon field with the potential, described above [115], but also with some quantity of dust. What will happen in such universe when the energy density of the pseudotachyon field tends to zero, while its pressure tends to infinity? In this case the deceleration also tends to infinity, while the energy density of the dust is finite and, hence, the universe should continue its expansion. However, if the universe continues the expansion the energy density of the pseudotachyon field becomes imaginary. Thus, we have some kind of a paradox [150]. The solution of this paradox was first found for the case of the anti-Chaplygin gas - perfect fluid with the equation of state

$$p = \frac{A}{\rho},\ A > 0,$$

which represents the simplest model, where the Big Brake singularity arises. The solution of the problem [131] consists in the fact the equation of state of this gas undergoes a transformation and it becomes the standard Chaplygin gas, but with a negative energy density. This solution was extended to the case of the pseudotachyon, which transforms itself into the quasitachyon with the Lagrangian

$$L = W(T)\sqrt{\dot{T}^2 + 1}. \tag{8.37}$$

Let us present in detail what happens with the pseudotachyon field when the universe in the presence of dust is running toward the future soft singularity. It behaves as

$$T(t) = T_s + \frac{2}{\sqrt{6H_S}}\sqrt{-t}, \tag{8.38}$$

where the value of the Hubble constant at the singularity H_S is found from the Friedmann equation for the universe filled with dust

$$H_S^2 = \frac{\rho_0}{a_S^3}, \tag{8.39}$$

where ρ_0 is a positive constant. To get the correct equation for the linearized perturbations of the pseudotachyon field in the vicinity of the singularity we use the Friedmann equation in the presence of both the pseudotachyon field and dust

$$\frac{\dot{a}^2}{a^2} = H_S^2 + \frac{W(T_0)}{\dot{T}^2 - 1}. \tag{8.40}$$

As a result we obtain the following equation for the linear perturbations of the pseudotachyon field (where as before we keep only the leading terms in the coefficients before $\ddot{\tilde{T}}, \dot{\tilde{T}}$ and $\tilde{T}$):

$$\ddot{\tilde{T}} - \frac{1}{2t}\dot{\tilde{T}} + \frac{B^2}{6H_S t}\tilde{T} = 0, \tag{8.41}$$

where

$$B^2 = \frac{W_{,TT}(T_S)}{W(T_S)} - \frac{W_{,T}^2(T_S)}{W^2(T_S)} + \frac{k^2}{a_S^2} > 0.$$

The solution of this equation is

$$\tilde{T}(t) = c_1 t^{3/4} J_{\frac{3}{2}}\left(\frac{B}{\sqrt{6H_S}} t^{\frac{1}{2}}\right) + c_2 t^{3/4} Y_{\frac{3}{2}}\left(\frac{B}{\sqrt{6H_S}} t^{\frac{1}{2}}\right). \tag{8.42}$$

Thus, we see both solutions of Eq. (8.41) are regular, and we can construct the creation and the annihilation operators and the Fock space. The basis functions in the vicinity of the singularity behave like

$$u = D + iF(-t)^{\frac{3}{2}}, \tag{8.43}$$

and, hence,

$$\frac{\dot{u}^*}{u^*} \sim \frac{iF(-t)^{\frac{1}{2}}}{D}. \tag{8.44}$$

On the other hand, it follows from Eq. (8.38) that

$$\frac{V(T_S)}{\sqrt{\dot{T}^2 - 1}} \sim \sqrt{-t}. \tag{8.45}$$

We obtain the vacuum wave function in the form

$$\Psi_0(\tilde{T}) \sim \exp(-C(-t)\tilde{T}^2). \tag{8.46}$$

We encounter the same situation which we have seen at the corner crossing: the dispersion of the Gaussian function tends to infinity at the crossing of the singularity. Nevertheless the situation looks more regular in the presence of dust. How can one interpret this fact? Perhaps, it is possible to think that the fact that the evolution at the crossing of the singularity is driven mainly by the dust makes the behaviour of the particle-like modes of the tachyon field more regular.

Chapter 9

Phantom divide line crossing and particles

In the section 7.1 we have already mentioned the phantom cosmology and the Big Rip singularity. In this chapter we consider a model with the scalar field with the cusped potential, where the phantom divide line crossing occurs. Here the particles are well defined in the vicinity of this crossing point.

9.1 Phantom fields and phantom divide line

As was mention before, many cosmological observations, such as SNe Ia [7] or WMAP [141], indicate that our universe is undergoing an accelerated expansion and consists of about 70% dark energy with negative pressure, responsible for this acceleration. To accelerate the expansion, the equation of state $p = w\rho$ of dark energy must satisfy $w < -\frac{1}{3}$. In the previous section we introduced one of the possible candidates for dark energy - the tachyon field. Other candidate for dark energy is phantom energy, satisfying the equation of state with $w < -1$. The Lagrangian density for phantom has a negative sign kinetic energy term, i.e.,

$$L = -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \,. \tag{9.1}$$

The existence of the phantom phase of cosmological evolution introduces some important theoretical problems. For example, the existence of phantom energy could cause the expansion of the universe to accelerate so quickly, and after finite amount of time it encounters the Big Rip singularity.

Another interesting problem arising in connection with the phantom energy is the crossing of the phantom divide line. The analysis of observation indicate the existence of the moment when the universe changes the value of the parameter w from the region $w < -1$ to $w > -1$. In such a cosmological evolution the stage of the superacceleration with $w < -1$ is a temporary one, and it is being substituted at some moment by the transition to the normal acceleration with $w > -1$. This hypothetic phenomenon is called "phantom divide line crossing". It can be described by models, including two scalar fields - a standard one and a phantom. More interesting option involves the consideration of the scalar field non-minimally coupled to gravity where such effect is also possible [151, 152]. In paper [153] rather general family of Lagrangians with the nontrivial kinetic term of the k-essence type [154] was studied from the point of view of the possibility of the phantom divide line crossing. It was shown that such a phenomenon can occur, but it is unstable with respect to perturbations or the corresponding trajectories have measure zero in the space of all possible evolutions.

It should be noted however that phantom field with negative kinetic term violates the strong energy condition $(\rho + 3p) > 0$, the null energy condition $\rho + p \geq 0$ and maybe physically unstable. But the phantom instability can be cured in extended theories of gravity [155].

9.2 Phantom divide line crossing and particles

In papers [156, 157] one more opportunity of phantom divide line crossing was considered: the cosmological evolution driven by a scalar field with a cusped potential. Remarkably, a passage through the point where the Hubble parameter achieves a maximum value implies the change of the sign of the kinetic term. Though a cosmological singularity is absent in these cases, this phenomenon is a close relative of those, considered in the preceding sections, because here we also find some transformation of matter properties induced by a change of geometry. In this aspect the phenomenon of the phantom divide line crossing in the model [156, 157] is analogous to the transformation between the tachyon and pseudotachyon field in the Born-Infeld model with the trigonometric potential considered earlier.

Consider the phantom scalar field with a negative kinetic term and the potential which has the following form

$$V(\phi) = \frac{V_0}{(1 + V_1\phi^{\frac{2}{3}})^2}. \tag{9.2}$$

The Klein-Gordon equation for the homogeneous part of the phantom scalar field has the form

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{4V_0V_1}{3(1 + V_1\phi^{\frac{2}{3}})^3\phi^{\frac{1}{3}}} = 0. \tag{9.3}$$

The Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = -\frac{\dot{\phi}^2}{2} + \frac{V_0}{(1 + V_1\phi^{\frac{2}{3}})^2}. \tag{9.4}$$

We are interested in a special solution of these equations, when at some moment (we can choose it as $t = 0_-$) the phantom scalar field and its time derivative tend to zero. Such a solution exists and it looks as follows

$$\phi(t) = \phi_0(-t)^{\frac{3}{2}}, \tag{9.5}$$

$$\frac{\dot{a}^2}{a^2} = \sqrt{V_0}, \tag{9.6}$$

where

$$\phi_0 = \left(-\frac{16}{9}V_0V_1\right)^{\frac{3}{4}},\ V_0 > 0,\ V_1 < 0. \tag{9.7}$$

The analysis of the equations of motion (9.5) and (9.6) shows [156, 157] that the smooth evolution of the universe compatible with the particular initial conditions chosen in such a way to provide this regime is possible if at $t = 0_+$ the phantom field transforms itself into the standard scalar field. This kind of the transition is indeed smooth because the kinetic term changes its sign, passing through the point when it is equal to zero.

To explain better what happens at this passage through the point when both the field and its time derivative vanish we can recall briefly a simple mechanical analogy [157]. Let us consider a one-dimensional problem of a classical point particle moving in the potential

$$V(x) = \frac{V_0}{(1 + x^{2/3})^2}, \tag{9.8}$$

where $V_0 > 0$. The equation of motion is

$$\ddot{x} - \frac{4V_0}{3(1 + x^{2/3})^3 x^{1/3}} = 0. \tag{9.9}$$

There are three types of possible motions, depending on the value of the energy E. If $E < V_0$, the particle cannot reach the top of the potential at the point $x = 0$. If $E > V_0$, the particle passes through the top of the hill with a non-vanishing velocity. The case $E = V_0$ is exceptional. In the vicinity of the point $x = 0$ the trajectory of the particle is

$$x(t) = C(t_0 - t)^{3/2}, \tag{9.10}$$

where

$$C = \pm \left(\frac{16V_0}{9}\right)^{3/4} \tag{9.11}$$

and $t \leq t_0$. Independently of the sign of C in Eq. (9.11) the signs of the particle coordinate x and its velocity $\dot{x}$ are opposite and hence, the particle can arrive in finite time to the point of the cusp of the potential at $x = 0$. Another solution reads as

$$x = C(t - t_0)^{3/2}, \tag{9.12}$$

where $t \geq t_0$. This solution describes the particle going away from the point $x = 0$. Thus, we can combine the branches of the solutions (9.10) and (9.12) in four different manners and there is no way to choose if the particle arriving to the point $x = 0$ should go back or should pass the cusp of the potential (9.8). It can stop at the top as well. To observe an analogy between this problem and the cosmological one we can try to introduce a friction term into the Newton equation (9.9)

$$\ddot{x} + \gamma\dot{x} - \frac{4V_0}{3(1 + x^{2/3})^3 x^{1/3}} = 0. \tag{9.13}$$

If the friction coefficient γ is a constant, one does not have a qualitative change with respect to the discussion above. However, if γ is

$$\gamma = 3\sqrt{\frac{\dot{x}^2}{2} + V(x)}. \tag{9.14}$$

then

$$\dot{\gamma} = -\frac{3}{2}\dot{x}^2 \tag{9.15}$$

and

$$\ddot{\gamma} = -3\ddot{x}\dot{x} \tag{9.16}$$

just like in the cosmological case, where the role of the friction coefficient is played by the Hubble parameter. The trajectory arriving to the cusp with a vanishing velocity is still described by the solution (9.10). Consider the particle coming to the cusp from the left ($C < 0$). It is easy to see that the value of $\dot{\gamma}$ at the moment t_0 tends to zero, while its second derivative $\ddot{\gamma}$ given by Eq. (9.16) is

$$\ddot{\gamma}(t_0) = \frac{9}{8}C^2 > 0. \tag{9.17}$$

Thus, it looks like the friction coefficient γ reaches its minimum value at $t = t_0$. Let us suppose that the particle is coming back to the left from the cusp and its motion is described by Eq. (9.12) with negative C. A simple check shows that in this case

$$\ddot{\gamma}(t_0) = -\frac{9}{8}C^2 < 0. \tag{9.18}$$

Thus, from the point of view of the subsequent evolution this point looks as a maximum for the function $\gamma(t)$. In fact, it simply means the second derivative of the friction coefficient has a jump at the point $t = t_0$. It is easy to check that if instead of choosing the motion to the left, we shall move forward our particle to the right from the cusp ($C > 0$), the sign of $\ddot{\gamma}(t_0)$ remains negative as in Eq. (9.18) and hence we have the jump of this second derivative again. If one would like to avoid this jump, one should try to change the sign in Eq. (9.16). To implement it in a self-consistent way one can substitute Eq. (9.14) by

$$\gamma = 3\sqrt{-\frac{\dot{x}^2}{2} + V(x)} \tag{9.19}$$

and Eq. (9.13) by

$$\ddot{x} + \gamma\dot{x} + \frac{4V_0}{3(1+x^{2/3})^3 x^{1/3}} = 0. \tag{9.20}$$

In fact, it is exactly that what happens automatically in cosmology, when we change the sign of the kinetic energy term for the scalar field, crossing the phantom divide line. Naturally, in cosmology the role of γ is played by the Hubble variable H. The jump of the second derivative of the friction coefficient γ corresponds to the divergence of the third time derivative of the Hubble variable, which represents some kind of a very soft cosmological singularity. Thus, when we change in a smooth way the sign of the kinetic term of the scalar field, it means that whenever possible we prefer the smoothness of the spacetime geometry to the conservation of the form of the equations of motion for the matter fields.

Now, as in the preceding sections, we write down the equation for linearized perturbations of the phantom field approaching the moment of the phantom divide line crossing. Using Eqs. (9.3) and (9.4), we obtain

$$\ddot{\tilde{\phi}} + \left(3\sqrt{-\frac{\dot{\phi}^2}{2} + \frac{V_0}{(1+V_1\phi^{\frac{2}{3}})^2}} - \frac{3\dot{\phi}^2}{2\sqrt{-\frac{\dot{\phi}^2}{2} + \frac{V_0}{(1+V_1\phi^{\frac{2}{3}})^2}}}\right)\dot{\tilde{\phi}}$$
$$+ \left(\frac{4V_0V_1}{9\phi^{\frac{4}{3}}\left(1+V_1\phi^{\frac{2}{3}}\right)^3} + \frac{8V_0V_1^2}{3\phi^{\frac{2}{3}}\left(1+V_1\phi^{\frac{2}{3}}\right)^4} - \frac{2V_0V_1\dot{\phi}}{\sqrt{-\frac{\dot{\phi}^2}{2} + \frac{V_0}{\left(1+V_1\phi^{\frac{2}{3}}\right)^2}}\,\phi^{\frac{1}{3}}\left(1+V_1\phi^{\frac{2}{3}}\right)^3} + \frac{k^2}{a^2}\right)\tilde{\phi} = 0. \tag{9.21}$$

Using the relations (9.6) and (9.7), we reduce the previous equation to the following simple form

$$\ddot{\tilde{\phi}} + 3\sqrt{V_0}\dot{\tilde{\phi}} + \frac{1}{4t^2}\tilde{\phi} = 0. \tag{9.22}$$

Here, as in all the preceding considerations we have omitted the subleading contributions to the coefficients at $\tilde{\phi}$ and its derivatives. The solution of this equation in the vicinity of $t = 0$ looks as

$$\tilde{\phi}(t) = c_1\sqrt{-t} + c_2\sqrt{-t}\ln(-t). \tag{9.23}$$

We see that both the independent solutions of Eq. (9.23) are non-singular at $t \to 0_-$. Moreover, both of them tends to zero, while their Wronskian is constant. Thus, we can try to construct the vacuum and the Fock space. In the case of the minimally coupled scalar field we can directly use the formula (7.14). Note that the scale factor at the cusp has a finite value. Thus, all possible interesting effects are connected with the behavior of basis functions. Let us introduce

$$u = A\sqrt{-t} + iB\sqrt{-t}\ln(-t). \tag{9.24}$$

In this case, in the vicinity of the cusp

$$\Psi_0(f) \sim \frac{1}{\sqrt{\sqrt{-t}\ln(-t)}}\exp\left(-\frac{A}{B(-t)\ln^2(-t)}f^2 + \frac{i}{2t}f^2\right). \tag{9.25}$$

We have that at $t \to 0_-$ the dispersion of the Gaussian function tends to zero and the function becomes the Dirac delta function. After the crossing of the cusp the dispersion becomes regular again. One can interpret this as for a moment the vacuum and the Fock space disappear and then their reappear once again, while the particles of the phantom field become particles of the standard scalar field or vice versa.

Chapter 10

Concluding Remarks

In this thesis we studied some aspects of Brans-Dicke theory and cosmological models based on fields with non-standard kinetic terms or potential.

In the Section 4 we showed that regular bounce solutions without any phantom field, even in the Einstein frame, can arise in Brans–Dicke theories containing fluids obeying the equation of state $p = \alpha\rho$ if $\frac{1}{4} \leq \alpha < 1$, and a Brans–Dicke parameter ω lying in the interval $-\frac{3}{2} \leq \omega \leq -\frac{4}{3}$, enlarging the parameter space in which such cosmological models can emerge in this class of theories. We analyzed in detail the radiative case with $\alpha = 1$. A bounce can be obtained if we choose the lower sign in Eqs. (4.10)-(4.11) for $-\frac{3}{2} < \omega < 0$. Moreover, for $-\frac{3}{2} < \omega \leq -\frac{4}{3}$ the bounce is regular with no curvature singularity, but for $-\frac{4}{3} < \omega < 0$ there is a singularity at $\eta = \eta_+$, even if the scale factor diverges at this point. In the case of $\omega = -\frac{4}{3}$ there is still no singularity if we choose the lower sign, and there is an initial singularity for the upper sign. The solutions Eqs. (4.12)-(4.13) with $\omega < -\frac{3}{2}$ have a similar behavior, but with a phantom field in the Einstein frame.

It is generally expected that the violation of the energy conditions is required in order to have classical bounce solutions, even in the nonminimal coupling case: in this situation, phantom fields would appear in the Einstein frame. We discussed this point in detail for the case of the radiative fluid in the Brans–Dicke theory (with a flat spatial sections), where we have shown that it is possible to obtain nonsingular solutions preserving the energy conditions even in the Einstein frame, and we have shown that this property holds for any Brans–Dicke theory in which $\frac{1}{4} \leq \alpha < 1$, and $-\frac{3}{2} \leq \omega \leq -\frac{4}{3}$. This generalization allows the possibility of constructing more involved and realistic regular bouncing solutions, in which the power spectrum of cosmological perturbations could be in accordance with present observations.

In the Section 5 we analyzed the cosmological solution for a perfect fluid in Brans-Dicke theory and showed that stiff matter ($p = \rho$) is a very particular solution. There are power law solution with the scale factor and the scalar field, respectively, proportional to $a \propto t^{1/2}$ and $\phi \propto t^{-1}$. Even though the matter content behaves as stiff matter, the cosmological evolution mimics a radiation dominated dynamics in General Relativity. Furthermore, the scalar field gives the effective gravitational strength, hence, gravity becomes stronger with the expansion of the universe. Eq. (5.14) shows that the scalar field is inversely proportional to the BD parameter ω. This condition is commonly understood as a sufficient condition for a well defined GR limit. However, we have shown that this is not the case for the power law solution (5.10).

The scalar cosmological perturbation also has interesting features. The velocity field for the stiff matter fluid has a growing mode U that is proportional to scale factor. This extra contribution produces new polynomial solutions for the density contrast $\delta = \delta\rho/\rho$, the fractional scalar field perturbation $\lambda = \delta\phi/\phi$ and the tensor perturbation $h = h_{kk}/a^2$. The homogeneous mode has four power solutions in cosmic time t^m with $m = -1, 0, 1/2, 1$. The first two are connected with the residual freedom of the synchronous gauge and the other two are the physical solutions corresponding to two growing modes. There is no decaying mode. The inhomogeneous mode related to the growing mode U goes as $t^{3/2} \propto a^3$, hence it is a steep growth if compared with the standard cosmological model.

It is well known in the literature that there are examples where the BD parameter scales as $\phi \sim 1/\sqrt{\omega}$ but the system does not approach a GR regime in the limit $\omega \to \infty$. Nevertheless, it is commonly expected to recover GR in this limit if $\phi \sim 1/\omega$ and the matter energy-momentum tensor has a nonzero trace. We have explicitly showed an exact BD solution with $\phi \sim 1/\omega$ and $T^{\mu}{}_{\mu} \neq 0$ that does not approach GR in the limit $\omega \to \infty$.

The Section 8 was devoted to the study of a particular cosmological model based on the tachyon field with a trigonometrical potential. Two peculiar effects distinguish this model. First, there are transformations between different kinds of Born-Infeld type fields—tachyons, pseudotachyons and quasitachyons. Second, the appearance of the future big brake singularity or, in the presence of dust, a more general type of soft future singularity. Here, we have considered the behavior of the perturbations of the Born-Infeld type fields for three differential equations. The simplest case is the passing through the point where both the potential and the kinetic term are equal to zero. We saw that in this case both solutions of the corresponding differential equation are regular, but when passing through the corner the vacuum state in some manner disappear (one can interpret the infinite dispersion in this way), and immediately after the crossing we again have a Fock space. The situation is different when the universe driven by the pseudotachyon field approaches the big brake singularity. Here, one of the solutions is singular and the particles do not exist. Strangely, if we add to the model some quantity of dustlike matter, the character of the singularity changes slightly, and the differential equation for the perturbations of the pseudotachyon field has two independent regular solutions. Thus, the particles exist, and the presence of dust works as a factor "normalizing" the passage through the singularity.

We have noticed analyzing the examples in Secs. (7.1), (8.2) and (9) that if a field drives the evolution toward some special points like singularities then describing the linear perturbations of this field, which serve as a tool for the definition of the vacuum state, Fock space, and particles, we stumble upon singular basis functions. In the case of the model including the tachyon field and dust the evolution through the soft singularity is driven mainly by dust and not by tachyon field. That is a plausible reason for the appearance of the well-defined basis functions for the perturbations of the tachyon field. But the analysis of the vacuum wave function gives us the same situation which we saw at the corner crossing: the dispersion of the Gaussian function tends to infinity at the crossing of the singularity.

The last Section 9 was devoted to the model with the scalar field with cusped potential. Here, a particular regime exists. If we choose the initial conditions in a special way, then the phantom scalar field can be transformed into the standard scalar field with the positive kinetic term. In other words, the phantom divide line crossing occurs. There are two regular solutions for the perturbations of the scalar field in the vicinity of the crossing point, and both of them tend to zero in the corresponding limit. The dispersion of the Gaussian function tends to zero at $t \to 0_-$ and the function becomes the Dirac delta function. After the crossing of the cusp the dispersion becomes regular again. One can interpret this as for a moment the vacuum and the Fock space disappear and then reappear once again, while the particles of the phantom field become particles of the standard scalar field or vice versa.

Bibliography

[1] C. M. Will, *Theory and Experiment in Gravitational Physics*, 2nd edn. (Cambridge University Press, 2018).

[2] R. D. Reasenberg et al., *Viking relativity experiment: Verification of signal retardation by solar gravity*, Astrophys.J.Lett. 234 (1979) L219.

[3] C. W. F. Everitt et al., *Gravity Probe B: Final Results of a Space Experiment to Test General Relativity*, Phys.Rev.Lett. 106 (2011) 221101.

[4] D. C. Rodrigues, V. Marra, A. del Popolo and Z. Davari, *Absence of a fundamental acceleration scale in galaxies*, Nat.Astron. 2 (2018) 668–672.

[5] J. M. Weisberg et al., *Timing Measurements of the Relativistic Binary Pulsar PSR B1913+16*, Astrophys.J. 722 (2010) 1030.

[6] B. P. Abbott et al., *Gravitational Waves and Gamma-Rays from a Binary Neutron Star Merger: GW170817 and GRB 170817A*, Astrophys.J.Lett. 848 (2017) 2, L13.

[7] A. G. Riess et al. (Supernova Search Team), *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, Astron.J. 116 (1998) 1009-1038.

[8] V. Sahni and A. A. Starobinsky, *The Case for a Positive Cosmological Lambda-term*, Int.J.Mod.Phys. D 9 (2000) 373-444.

[9] E. J. Copeland, M. Sami, and S. Tsujikawa, *Dynamics of dark energy*, Int.J.Mod.Phys. D 15 (2006) 1753-1936.

[10] C. Brans and R. H. Dicke, *Mach's Principle and a Relativistic Theory of Gravitation*, Phys.Rev. 124 (1961) 925–935.

[11] T. Kobayashi, *Horndeski theory and beyond: a review*, Rept.Prog.Phys. 82 (2019) 8, 086901.

[12] P. A. M. Dirac, *A new basis for cosmology*, Proc.R.Soc.L. A 165 (1938) 199.

[13] P. Jordan, *Zur empirischen Kosmologie*, Naturwiss 26 (1938) 417-421.

[14] F. Wu and X. Chen, *Cosmic microwave background with Brans-Dicke gravity. II. Constraints with the WMAP and SDSS data*, Phys.Rev. D. 82 (2010) 083003.

[15] A. Avilez and C. Skordis, *Cosmological Constraints on Brans-Dicke Theory*, Phys.Rev.Lett. 113 (2014) 011101.

[16] M. J. Duff, R. R. Khuri, and J. X. Lu, *String Solitons*, Phys.Rept. 259 (1995) 213-32.

[17] S. Alexander, J. D. Barrow and J. Magueijo, *Turning on gravity with the Higgs mechanism*, Class. Quantum Grav. 33 (2016) 14LT01.

[18] D. A. Tretyakova, B. N. Latosh and S. O. Alexeyev, *Wormholes and Naked Singularities in Brans-Dicke cosmology*, Class. Quantum Grav. 32 (2015) 185002.

[19] O. Hrycyna and M. Szydlowski, *Dynamics and cosmological constraints on Brans-Dicke cosmology*, Phys.Rev. D 88 (2013) 064018.

[20] C. M. Will and N. Yunes, *Testing alternative theories of gravity using LISA*, Class.Quantum Grav. 21 (2004) 4367.

[21] G. Kofinas, E. Papantonopoulos and E. N. Saridakis, *Modified Brans-Dicke cosmology with matter-scalar field interaction*, Class.Quantum Grav. 33 (2016) 155004.

[22] G. Papagiannopoulos, J. D. Barrow, S. Basilakos, A. Giacomini and A. Paliathanasis, *Dynamical symmetries in Brans-Dicke cosmology*, Phys.Rev. D 95 (2017) 024021.

[23] D. Alonso, E. Bellini, P. G. Ferreira and M. Zumalacarregui, *Observational future of cosmological scalar-tensor theories*, Phys.Rev. D 95 (2017) 063502.

[24] N. Roy and N. Banerjee, *Generalized Brans-Dicke theory: A dynamical systems analysis*, Phys.Rev. D 95 (2017) 064048.

[25] J.-M. Gerard and I. Mahara, *Asymptotic solutions to a tensor-scalar theory of gravity*, Phys.Lett. B 346 (1995) 35.

[26] J. D. Barrow and D. J. Shaw, *Some late-time asymptotics of general scalar-tensor cosmologies*, Class.Quant.Grav. 25 (2008) 085012.

[27] R. H. Dicke, *Mach's principle and invariance under transformation of units*, Phys.Rev. 125 (1962) 2163.

[28] V. Faraoni, Sh. Nadeau, *The (pseudo)issue of the conformal frame revisited*, Phys.Rev. D 75 (2007) 023501.

[29] Salvatore Capozziello, S. Nojiri, S. D. Odintsov, A. Troisi, *Cosmological viability of f(R)-gravity as an ideal fluid and its compatibility with a matter dominated phase*, Phys.Lett. B 639 (2006) 135.

[30] E. Frion, C. R. Almeida, *Affine quantization of the Brans-Dicke theory: Smooth bouncing and the equivalence between the Einstein and Jordan frames*, Phys.Rev.D 99 (2019) 2, 023524.

[31] C. F. Steinwachs, A. Yu. Kamenshchik, *One-loop divergences for gravity nonminimally coupled to a multiplet of scalar fields: Calculation in the Jordan frame. I. The main results*, Phys.Rev.D 84 (2011) 024026.

[32] Narayan Banerjee and Barun Majumder, *A question mark on the equivalence of Einstein and Jordan frames*, Phys.Lett. B 754 (2016) 129-134.

[33] V. Faraoni, E. Gunzig, P. Nardone, *Conformal transformations in classical gravitational theories and in cosmology*, Fund.Cosmic Phys. 20 (1999) 121.

[34] V. Faraoni, *Cosmology in Scalar-Tensor Gravity* (Kluwer Academic Publishers, 2004).

[35] L. E. Gurevich, A. M. Finkelstein and V. A. Ruban, *On the problem of the initial state in the isotropic scalar-tensor cosmology of Brans-Dicke*, Astrophys.SpaceSci. 22 (1973) 231.

[36] Aghanim N.,et al., *Planck 2018 results. VI. Cosmological parameters*, Astron.Astrophys. 641, A6 (2020).

[37] Oliver F. Piattella, *Lecture Notes in Cosmology* (Springer, 2018).

[38] A. A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, JETP Lett. 30 (1979) 682-685. [*Спектр реликтового гравитационного излучения и начальное состояние Вселенной*, Pisma Zh.Eksp.Teor.Fiz. 30,719 (1979)].

[39] A. H. Guth, *The inflationary universe: A possible solution to the horizon and flatness problems*, Phys.Rev. D 23 (1981) 347-356.

[40] A. D. Linde, *A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems*, Phys.Lett. B 108 (1982) 389-393.

[41] J. D. Barrow and A. R. Liddle, *Can inflation be falsified?* Gen.Rel.Grav. 29 (1997) 1503.

[42] A. Ijjas, P. J. Steinhardt, and A. Loeb, *Inflationary paradigm in trouble after Planck2013*, Phys.Lett. B723 (2013) 261.

[43] A. Linde, *Inflationary cosmology after Planck 2013*, arXiv:1402.0526v2 (2014).

[44] D. Battefeld, Patrick Peter, *A critical review of classical bouncing cosmologies*, Phys.Rept. 571 (2015) 1-66.

[45] M. Novello and S. E. P. Bergliaffa, *Bouncing cosmologies*, Phys.Rept. 463 (2008) 127-213.

[46] A. Ijjas , P. J. Steinhardt, *Bouncing cosmology made simple*, Class.Quant.Grav. 35 (2018) 13, 135004.

[47] P. Peter and N. Pinto-Neto, *Primordial perturbations in a non singular bouncing universe model*, Phys.Rev. D65 (2002) 023513.

[48] V. Melnikov and S. Orlov, *Nonsingular cosmology as a quantum vacuum effect*, Phys.Lett. A70 (1979) 263.

[49] M. Novello, L. A. R. Oliveira, J. M. Salim, E. Elbaz, *Geometrized instantons and the creation of the universe*, Int.J. of Mod.Phys. D1 (1993) 641-677.

[50] V. Ts. Gurovich, *The nonlinear correction in the lagrange density of the gravitational field, and cosmological solutions with no singularity*, Sov.Phys.-Dokl. 15 (1971) 1105;

[51] H. Nariai, *On the removal of initial singularity in a Big-Bang universe in terms of a renormalized theory of gravitation. I: Examination of the present status and a new approach*, Prog.Theor.Phys. 46 (1971) 433;

[52] H. Nariai and K. Tomita, *On the removal of initial singularity in a Big-Bang universe in terms of a renormalized theory of gravitation. II: Criteria for obtaining a physically reasonable model*, Prog.Theor.Phys. 46 (1971) 776;

[53] R. Brandenberger, V. Mukhanov and A. Sornborger, *Cosmological theory without singularities*, Phys.Rev. D48 (1993) 1629.

[54] S. Mukherji and M. Peloso, *Bouncing and cyclic universes from brane models*, Phys.Lett. B 547 (2002) 297-305.

[55] A. A. Starobinsky, *On a nonsingular isotropic cosmological model*, Pis'ma Astron.Zh. 4 (1978) 155-159.

[56] V. A. Belinsky, I. M. Khalatnikov, L. P. Grishchuk, Ya. B. Zeldovich, *Inflationary stages in cosmological models with a scalar field*, Phys.Lett. B 155 (1985) 232-236.

[57] B. Boisseau, H. Giacomini, D. Polarski, A. A. Starobinsky, *Bouncing universes in scalar-tensor gravity models admitting negative potentials*, JCAP 07 (2015) 002.

[58] B. Boisseau, H. Giacomini, D. Polarski, *Scalar field cosmologies with inverted potentials*, JCAP 10 (2015) 033.

[59] G. Brando, J. C. Fabris, F. T. Falciano, and O. Galkina, *Stiff matter solution in Brans–Dicke theory and the general relativity limit*, Int.J.Mod.Phys. D 28 (2019) 12, 1950156.

[60] A. A. Starobinsky, *Can the effective gravitational constant become negative?* Sov.Astron.Lett. 7 (1981) 36-38 [*Может ли эффективная гравитационная постоянная стать отрицательной?* Письма в АЖ 7 (1981) 67-72].

[61] K. A. Bronnikov, *Scalar-tensor gravity and conformal continuations*, J.Math.Phys. 43 (2002) 6096-6115.

[62] L. Parker and S. A. Fulling, Quantied matter fields and the avoidance of singularities in General Relativity, Phys. Rev. D 7 (1973) 2357-2374.

[63] A. A. Starobinsky, *On a nonsingular isotropic cosmological model*, Sov.Astron.Lett. 4 (1978) 82 [*Об одной несингулярной изотропной космологической модели*, Письма в АЖ 4 (1978) 155-159].

[64] A. Yu. Kamenshchik, I. M. Khalatnikov, and A. V. Toporensky, *Simplest cosmological model with the scalar field*, Int.J.Mod.Phys.D 6 (1997) 673-692.

[65] A. Yu. Kamenshchik, I. M. Khalatnikov, S. V. Savchenko, and A. V. Toporensky, *Topological entropy for some isotropic cosmological models*, Phys.Rev. D 59 (1999) 123516.

[66] A. Yu. Kamenshchik, E. O. Pozdeeva, S. Vernov, A. Tronconi, and G. Venturi, *Transformations between Jordan and Einstein frames: Bounces, antigravity, and crossing singularities*, Phys.Rev. D 94 (2016) 6, 063510.

[67] J. P. Baptista, J. C. Fabris, and S. V. B. Gonçalves, *Density perturbations in the Brans-Dicke theory*, Astrophys.Space Sci. 246 (1996) 315.

[68] J. C. Fabris, A. B. Batista, and J. P. Baptista, *Density perturbations derived from general cosmological solutions of Brans-Dicke theory*, C.R.Acad.Sci. 309 (1989) 791.

[69] S. D. P. Vitenti and N. Pinto-Neto, *Large adiabatic scalar perturbations in a regular bouncing universe*, Phys.Rev. D 85 (2012) 023524.

[70] S. Yazadjiev, *Solution generating in scalar-tensor theories with a massless scalar field and stiff perfect fluid as a source*, Phys.Rev.D 65 (2002) 084023.

[71] D. J. Holden and D. Wands, *Self-similar cosmological solutions with a non-minimallycoupled scalar field*, Class.Quantum Grav. 15 (1998) 3271.

[72] J. D. Barrow and P. Parsons, *Behavior of cosmological models with varying G*, Phys.Rev. D 55 (1997) 1906.

[73] V. Faraoni, *Scalar field mass in generalized gravity*, Class.Quantum Grav. 26 (2009) 145014.

[74] S. Weinberg, Gravitation and Cosmology (John Wiley & Sons, 1972).

[75] L. Jarv, P. Kuusk, M. Saal and O. Vilson, *Transformation properties and general relativity regime in scalar–tensor theories*, Class.Quantum Grav. 32 (2015) 235013.

[76] H. Nariai, On the Green's function in an expanding universe and is role in the problem of Mach's principle, Prog.Theor.Phys. 40 (1968) 49.

[77] A. Banerjee, N. Banerjee and N. O. Santos, *Anisotropic cosmological model in Nordtvedt's scalar–tensor theory of gravitation*, J.Math.Phys. 26 (1985) 3125.

[78] N. Banerjee, *Static cylindrically symmetric charge distribution in general scalar tensor theory of gravitation*, Acta Phys.Pol. B 17 (1986) 131.

[79] N. Banerjee and S. Sen, *Does Brans-Dicke theory always yield general relativity in the infinite limit?* Phys.Rev. D 56 (1997) 1334.

[80] J. O'Hanlon and B. Tupper, *Vacuum-field solutions in the Brans-Dicke theory*, Nuovo Cimento A 7 (1972) 305.

[81] T. Matsuda, *On the gravitational collapse in Brans-Dicke theory of gravity*, Prog.Theor.Phys. 47 (1972) 738.

[82] C. Romero and A. Barros, *Does the Brans-Dicke theory of gravity go over to general relativity when* $\omega \to \infty$*?* Phys.Lett. A 173 (1993) 243.

[83] C. Romero and A. Barros, *Brans-Dicke vacuum solutions and the cosmological constant: A Qualitative analysis*, Gen.Relativ.Gravit. 25 (1993) 491.

[84] F. M. Paiva, M. Rebouças and M. MacCallum, *On limits of spacetimes - a coordinate-free approach*, Class.Quantum Grav. 10 (1993) 1165.

[85] F. M. Paiva and C. Romero, *The limits of Brans-Dicke spacetimes: a coordinate-free approach*, Gen.Relativ.Gravit. 25 (1993) 1305.

[86] M. A. Scheel, S. L. Shapiro and S. A. Teukolsky, *Collapse to black holes in Brans-Dicke Theory: II. Comparison with General Relativity*, Phys.Rev. D 51 (1995) 4236.

[87] V. Faraoni, *Illusions of general relativity in Brans-Dicke gravity*, Phys.Rev. D 59 (1999) 084021.

[88] B. Chauvineau, *On the limit of Brans–Dicke theory when* $\omega \to \infty$, Class.Quantum Grav. 20 (2003) 2617.

[89] J. P. Baptista, J. C. Fabris and S. V. B. Gonçalves, *Density perturbations in the Brans-Dicke theory*, Astrophys.Spc.Sci. 246 (1996) 315.

[90] P. J. E. Peebles, The Large Scale Structure of the Universe (Princeton University Press, New Jersey, 1980).

[91] L. Amendola and S. Tsujikawa, Dark Energy: Theory and Observations (Cambridge University Press, 2015).

[92] S. Weinberg, Cosmology (Oxford University Press, 2008).

[93] G. Brando, F. T. Falciano and L. F. Guimarães, *Spacetime singularities in generalized Brans-Dicke theories*, Phys.Rev. D 98 (2018) 044027.

[94] G. Sansoni and R. Conti, Non-linear Differential Equations (Pergamon Oxford, 1964).

[95] O. I. Bogoyavlensky, Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics (Springer Berlin, 1985).

[96] J. C. Fabris and J. Tossa, *Qualitative study of multidimensional cosmological models*, Phys.Lett. B 296 (1992) 307–310.

[97] V. Belinski, M. Henneaux, *The Cosmological Singularity* (Cambridge University Press, Cambridge, 2017).

[98] A. Friedman, *On the curvature of space*, Z.Phys. 10 (1922) 377-386, Gen.Rel.Grav. 31 (1999) 1991-2000.

[99] V. A. Belinsky, I. M. Khalatnikov, *On the nature of the singularities in the general solutions of the gravitational equations*, Sov.Phys. JETP 29 (1969) 911 [*К вопросу о характере особенностей в общих решениях уравнений гравитации*, Zh.Eksp.Teor.Fiz. 56 (1969) 1700].

[100] V. A. Belinsky, E. M. Lifshitz, I. M. Khalatnikov, *Oscillatory approach to the singular point in relativistic cosmology*, Sov.Phys.Usp. 13 (1971) 745 [*Колебательный режим приближения к особой точке в релятивистской космологии*, Usp.Fiz.Nauk 102 (1970) 463]

[101] V. A. Belinsky, E. M. Lifshitz, I. M. Khalatnikov, *A general solution of the Einstein equations with a time singularity*, Adv.Phys. 31 (1982) 639-667.

[102] C. W. Misner, *Mixmaster Universe*, Phys.Rev.Lett. 22 (1969) 1071-1074.

[103] E. M. Lifshitz, I. M. Lifshitz, I. M. Khalatnikov, A*symptotic analysis of oscillatory mode of approach to a singularity in homogeneous cosmological models*, Sov.Phys.JETP 32 (1971) 173 [*Асимптотический анализ колебательного режима приближения к особой точке в однородных космологических моделях*, Zh.Eksp.Teor.Fiz 59 (1970) 322].

[104] I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, L. N. Shchur , Ya. G. Sinai, J.Stat.Phys. 38 (1985) 97.

[105] R. Penrose, *Gravitational collapse and space-time singularities*, Phys.Rev.Lett. 14 (1965) 57-59.

[106] S.W. Hawking, *Properties of expanding universes*, PhD Thesis, University of Cambridge (1966).

[107] A. A. Starobinsky, Future and origin of our universe: Modern view, Grav.Cosmol. 6 (2000) 157-163.

[108] R. R. Caldwell, M. Kamionkowski, and N. N. Weinberg, *Phantom energy: Dark energy with* $w < -1$ *causes a cosmic doomsday*, Phys.Rev.Lett. 91 (2003) 071301.

[109] R. R. Caldwell, *A Phantom Menace? Cosmological consequences of a dark energy component with super-negative equation of state*, Phys.Lett. B 545 (2002) 23-29.

[110] J. D. Barrow, G. J. Galloway and F. Tipler, *The closed-universe recollapse conjecture*, Mon.Not.Roy.Astron. Soc. 223 (1986) 835.

[111] M. P. Dabrowski, T. Denkiewicz and M. A. Hendry, *How far is it to a sudden future singularity of pressure?* Phys.Rev. D 75 (2007) 123524.

[112] T. Denkiewicz, *Observational Constraints on Finite Scale Factor Singularities*, JCAP 07 (2012) 036.

[113] T. Denkiewicz, M. P. Dabrowski, H. Ghodsi and M. A. Hendry, *Cosmological tests of sudden future singularities*, Phys.Rev. D 85 (2012) 083527.

[114] Z. Keresztes, L. A. Gergely, V. Gorini, U. Moschella and A. Yu. Kamenshchik, *Tachyon cosmology, supernovae data, and the big brake singularity*, Phys.Rev. D 79 (2009) 083504.

[115] V. Gorini, A. Y. Kamenshchik, U. Moschella and V. Pasquier, *Tachyons, scalar fields and cosmology*, Phys.Rev. D 69 (2004) 123512.

[116] P. Tretyakov, A. Toporensky, Yu. Shtanov and V. Sahni, *Quantum effects, soft singularities and the fate of the universe in a braneworld cosmology*, Class.Quantum Grav. 23 (2006) 3259-3274 .

[117] J. D. Barrow, A. B. Batista, G. Dito, J. C. Fabris and S. Houndjo, *Sudden singularities survive massive quantum particle production*, Phys.Rev. D 84 (2011) 123518.

[118] J. de Haro, J. Amoros and E. Elizalde, *Sudden singularities in semiclassical gravity*, Phys.Rev. D 85 (2012) 123527.

[119] M. P. Dabrowski, C. Kiefer and B. Sandhofer, *Quantum phantom cosmology*, Phys.Rev. D 74 (2006) 044022.

[120] A. Yu. Kamenshchik, C. Kiefer, and B. Sandhöfer, *Quantum cosmology with a big-brake singularity*, Phys.Rev. D 76 (2007) 064032.

[121] M. Bouhmadi-Lopez, C. Kiefer, B. Sandhofer, and P. V. Moniz, *Quantum fate of singularities in a dark-energy dominated universe*, Phys.Rev. D 79 (2009) 124035.

[122] A. Y. Kamenshchik and S. Manti, *Scalar field potentials for closed and open cosmological models*, Phys.Rev. D 85 (2012) 123518.

[123] E. M. Barboza, Jr. and N. A. Lemos, *Does the Big Rip survive quantization?* Gen.Rel.Grav. 38 (2006) 1609-1622.

[124] H. Calderon and W. A. Hiscock, *Quantum fields and "Big Rip" expansion singularities*, Class.Quant. Grav. 22 (2005) L23-L26.

[125] Y. Shtanov and V. Sahni, *New cosmological singularities in braneworld models*, Class.Quant.Grav. 19 (2002) L101-L107.

[126] L. Fernández-Jambrina and R. Lazkoz, *Geodesic behavior of sudden future singularities*, Phys.Rev. D 70 (2004) 121503.

[127] Z. Keresztes, L. A. Gergely, A. Y. Kamenshchik, V. Gorini, and D. Polarski, *Will the tachyonic Universe survive the Big Brake?* Phys.Rev. D 82 (2010) 123534.

[128] F. J. Tipler, *Singularities in conformally flat spacetimes*, Phys.Lett. A 64 (1977) 8-10.

[129] V. N. Lukash, E. V. Mikheeva, and V. N. Strokov, *The early universe and cosmogenesis*, Phys.Usp. 55 (2012) 204.

[130] V. N. Lukash, E. V. Mikheeva, and V. N. Strokov, *Generation of cosmological flows in General Relativity (features and properties of integrable singularities)*, Phys.Usp. 55 (2012) 831-837.

[131] Z. Keresztes, L. A. Gergely, A. Yu. Kamenshchik, V. Gorini, and D. Polarski, *Soft singularity crossing and transformation of matter properties*, Phys.Rev. D 88 (2013) 023535.

[132] A. Y. Kamenshchik, *Singularity crossing, transformation of matter properties and the problem of parametrization in field theories*, Found.Phys. 48 (2018) 10, 1159-1176.

[133] A. Yu. Kamenshchik, *Quantum cosmology and late-time singularities*, Class.Quant.Grav. 30 (2013) 173001.

[134] Sh. Nojiri, S. D. Odintsov and S. Tsujikawa, *Properties of singularities in the (phantom) dark energy universe*, Phys.Rev. D 71 (2005) 063004.

[135] A. Sen, *Rolling tachyon*, JHEP 04 (2002) 048.

[136] A.A. Starobinsky, *A new type of isotropic cosmological models without singularity*, Phys.Lett. B91 (1980) 99-102.

[137] M.R. Setare and E.N. Saridakis, *Non-minimally coupled canonical, phantom and quintom models of holographic dark energy*, Phys.Lett. B 671 (2009) 331.

[138] V. Sahni and Yu. Shtanov, *Did the universe loiter at high redshifts?* Phys.Rev. D 71 (2005) 084018.

[139] M. P. Dabrowski and T. Denkiewicz, *Barotropic index w-singularities in cosmology*, Phys.Rev. D 79 (2009) 06352.

[140] M. P. Dabrowski and K. Marosek, *Regularizing cosmological singularities by varying physical constants*, JCAP 02 (2013) 012.

[141] C. L. Bennett, et al., *First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results,* Astrophys.J.Suppl. 148 (2003) 1-27.

[142] B. S. DeWitt, *Quantum field theory in curved spacetime*, Phys.Rept. 19 (1975) 295-357.

[143] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge, England, 1984).

[144] L. E. Parker and D. Toms, Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity (Cambridge University Press, Cambridge, England, 2009).

[145] T. S. Bunch and P. C. W. Davies, *Quantum field theory in de Sitter space: renormalization by point-splitting*, Proc.Roy.Soc.Lond. A 360 (1978) 117-134.

[146] A. Feinstein, *Power-law inflation from the rolling tachyon*, Phys.Rev. D 66 (2002) 063511.

[147] T. Padmanabhan, *Accelerated expansion of the universe driven by tachyonic matter*, Phys.Rev. D 66 (2002) 021301.

[148] A. V. Frolov, L. Kofman, and A. A. Starobinsky, *Prospects and problems of tachyon matter cosmology*, Phys.Lett. B 545 (2002) 8-16.

[149] M. Born and L. Infeld, *Foundations of the new field theory*, Proc.Roy.Soc.Lond. A 144 (1934) 852, 425-451.

[150] Z. Keresztes, L. A. Gergely, and A. Y. Kamenshchik, *Paradox of soft singularity crossing and its resolution by distributional cosmological quantities*, Phys.Rev. D 86 (2012) 063522.

[151] B. Boisseau, D. Esposito-Farese, D. Polarski, and A. A. Starobinsky, *Reconstruction of a scalar-tensor theory of gravity in an accelerating universe*, Phys.Rev.Lett. 85 (2000) 2236.

[152] R. Gannouji, D. Polarski, A. Ranquet, and A. A. Starobinsky, *Scalar-tensor models of normal and phantom dark energy*, JCAP 09 (2006) 016.

[153] A. Vikman, *Can dark energy evolve to the phantom?* Phys.Rev. D 71 (2005) 023515.

[154] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, *Essentials of k-essence*, Phys.Rev. D 63 (2001) 103510.

[155] F. C. Carvalho and A. Saa, *Non-minimal coupling, exponential potentials and the* $\omega < -1$ *regime of dark energy*, Phys.Rev.D 70 (2004) 087302.

[156] A. A. Andrianov, F. Cannata, and A. Yu. Kamenshchik, *Smooth dynamical crossing of the phantom divide line of a scalar field in simple cosmological models*, Phys.Rev. D 72 (2005) 043531.

[157] F. Cannata and A. Yu. Kamenshchik, *Networks of cosmological histories, crossing of the phantom divide line and potentials with cusps*, Int.J.Mod.Phys. D 16 (2007) 1683-1704.