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Starobinsky inflation and the order reduction technique

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A presente tese "*Starobinsky inflation and the order reduction technique*" foi submetida no ano de 2022 ao PPGCosmo por *Waleska Priscylla Florencio de Medeiros* como parte dos requisitos para a obtenção do título de Doutor em Astronomia e Física.

Caso esta tese venha a ser aprovada, esta folha deverá ser substituída pela correspondente de aprovação.

To my parents and my brother

Abstract

The order reduction technique (ORT) is an iterative method of solution of higher order differential equations. It consists of treating the higher order terms perturbatively so that the lower order in the order reduction must be chosen according to which regime of solution the method is going to reproduce. In some cases, the presence of solutions that do not have physical behavior is observed, mainly associated with particularly higher order differential equations. Fortunately, as it is known in the literature, the order reduction method presents fewer solutions, and with that, one of the intentions of the technique is to make it easier to select the solutions which present a good physical behavior. However, it must be emphasized that one disadvantage of the method is that there could be some physical solutions that the order reduction will not detect.

The ORT is applied to the following cases:

1. The study of the dynamics of the motion of a charged particle.
2. The harmonic oscillator.
3. The inflationary paradigm of Starobinsky.

We show that in the case of the examples cited above, the ORT as an iterative perturbative method does not show convergence in the oscillating regime of a weak coupling limit. This regime is excluded by the order reduction. In addition, the method shows good convergence in the strong coupling regime, non-oscillating which slowly approaches equilibrium.

The main results discussed are based on the work [1].

Keywords: order reduction technique, harmonic oscillator, Starobinsky inflation.

Resumo

A técnica de redução de ordem (TRO) é um método iterativo de solução de equações diferenciais de ordem superior. Consiste em tratar de forma perturbativa os termos de ordem superior de modo que a ordem inferior na redução de ordem deve ser escolhida de acordo com qual regime de solução o método deve reproduzir. Em alguns casos, a presença de soluções que não possuem comportamento físico é observada, principalmente associada à equações diferenciais particulares de ordem superior. Felizmente, como é conhecido na literatura, o método de redução de ordem apresenta um número menor de soluções, e com isso, uma das intenções da técnica é facilitar a seleção das soluções que apresentam um bom comportamento físico. No entanto, deve ser enfatizado que uma desvantagem do método é que pode haver algumas soluções físicas que a redução de ordem não irá detectar.

A TRO é aplicada nos seguintes casos:

1. O estudo da dinâmica do movimento de uma partícula carregada.
2. O oscilador harmônico.
3. O paradigma inflacionário de Starobinsky.

Mostramos que no caso dos exemplos citados acima, a TRO como um método perturbativo iterativo não apresenta convergência no regime oscilante do limite de acoplamento fraco. Este regime está excluído pelo método de redução de ordem. Além disso, o método mostra boa convergência no regime de acoplamento forte, não oscilante que se aproxima lentamente do equilíbrio.

Os principais resultados são baseados no trabalho [1].

Palavras-chaves: técnica de redução de ordem. oscilador harmônico. inflação de Starobinsky.

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Contents

CHAPTER 1.

Introduction

"Life is like riding a bicycle.
To keep your balance, you must keep moving."
— A. Einstein.

Our current understanding of the universe is based on the standard model of cosmology, the so-called ΛCDM model. In this model, the 95% of the energy content of the universe is constituted by cold dark matter (CDM) and dark energy (DE) in the form of the cosmological constant Λ. Dark matter is cold, without electromagnetic interaction, and non-baryonic, and its existence is indicated by several astronomical measurements as a consequence of its gravitational interaction. There are a lot of candidates for dark matter, one popular among them are the particles beyond the standard model [2]. Dark energy is the name given to a type of anti-gravity energy that could be responsible for the accelerated expansion of the universe; the cosmological constant is one of the candidates for DE.

Standard cosmology also presents issues such as the flatness and the horizon problems. The universe must be flat now and flatter at the beginning, since $\Omega_k \propto 1/(Ha)^2$. The latest Planck data constrain the present curvature density as $\Omega_{k,0} = 0.0007 \pm 0.0019$ [3]. Theoretically, we have $\Omega_k(10^{-43}\text{ sec}) \leq O(10^{-60})$ at the Planck epoch and $\Omega_k(1\text{ sec}) \leq O(10^{-16})$ at the epoch of the primordial nucleosynthesis. So, the flatness problem consists in a fine-tuning of the initial conditions for the spatial curvature. For example, Ω_k must be determined with about 60 significant digits of precision at the Planck scale.

The horizon problem arises from *Cosmic Microwave Background Radiation* (CMBR) observation. The Hot Big Bang cosmological model does not explain why regions that apparently have never been in thermal contact present similar physical properties [4]. These problems can be solved, or at least alleviated if the universe went through an accelerated epoch in its beginnings. This epoch is called inflation [4], [5], [6], [7], [8] and [9]. Several models of inflation emerged since the end of the 1970's [10]. In particular, this thesis focuses on the model that arises from quadratic corrections to Einstein's general relativity: the Starobinsky inflationary model [11]. The main motivation for this theory is that it arises naturally as a quantum correction in a consistent model of semi-classical gravity, i.e. in a scenario where quantum matter fields are considered in a classical gravitational background, [12]. The action for this theory is:

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{PL}^2}{2}\left\{R + \beta R^2 + \alpha\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right]\right\} \tag{1.1}$$

include Starobinsky's model [11]. The value of $\beta \approx 1.305 \times 10^9\ M_{PL}^{-2}$ can be inferred from observation [13], [14], [15].

Different inflationary models are described by essentially two parameters: r, the tensor-to-scalar ratio, and n_S, the scalar index. They are obtained from the amplitudes of the power spectrum (scalar and tensor ones) of the primordial fluctuations produced during inflation. Different inflationary models differ in their predictions of these parameters [16], which can be constrained from observation of the CMBR.

The most precise observational tests for inflation have been performed with the help of the Planck [13] satellite. With better predictive power for the future, the *Probe of Inflation and Cosmic Origins* (PICO) aims to determine the energy scale of inflation [16] and the *Cosmic Origins Explorer* (COrE) designed to detect the primordial gravitational waves generated during the inflation period [17]. According to the latest CMBR observations [13], [16], [17] the Starobinsky model is the one that best fits the amplitude of the scalar-tensor ratio.

Metric variations in the action (1.1) result in partial differential field equations of order-4. Therefore, the Starobinsky model contains a differential partial equation with higher–than–second order time derivatives.

Higher derivatives were also studied in the context of electrodynamics by Lorentz and Abraham [18]. In particular, in the study of the dynamics of a charged particle in an electromagnetic field, it is considered by Lorentz a simple model in which the electrons are bound elastically and the system treated as a charged harmonic oscillator. Making an analogy with Larmor's work on the equality of energy loss radiated by a harmonic oscillator with a damping force, Lorentz found an additional term that arises due to its accelerated motion [19],

$$\mathbf{f_{self}} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{a}}. \tag{1.2}$$

Here q and m are the charge and mass of the particle, respectively, c is the speed of light, and $\dot{a}$ is the first order derivative of the particle's acceleration with respect to the time. This term arises through the balance between mechanical energy and energy radiated to infinity from the particle, which in turn acts back on it as a back-reaction force and alters the electron's motion, this term is called self-force.

After that, Abraham considered, as a model for the electron, a uniformly charged rigid sphere [20]. However, in 1904, Abraham realized the inconsistency of his model with the *Lorentz transformations* (LT). Right after these transformations were better established by Einstein, Abraham [18] abandoned his rigid electron by an electron whose shape change under LT and gives a relativistic generalization for the self-force term found by

Lorentz (1.2) as

$$\frac{2}{3}\frac{q^2}{c^3}\left(g_{\mu\nu} + u_\mu u_\nu\right)\dot{a}^\nu. \tag{1.3}$$

Where $g_{\mu\nu}$ is the metric tensor, $u^\mu = \mathrm{d}z/\mathrm{d}\tau$ the four-velocity, $a^\mu = \mathrm{d}u^\mu/\mathrm{d}\tau$ the four-acceleration and $\dot{a}^\mu = \mathrm{d}a^\mu/\mathrm{d}\tau$ the first derivative of the four-acceleration of the particle along the world line which is described by the relation $z(\tau)$, where τ represents the proper-time.

Nevertheless, the Lorentz-Abraham model does not hold in the case where the radius of the electron tends to zero, being the self-energy divergent. Dirac solved this problem, generalizing the Lorentz-Abraham model [21]. Dirac showed that the field responsible for the radiation can be better conveniently written as a combination of the retarded and advanced electromagnetic fields. Also, he considered a thin tube surrounding the electron world line in space-time to calculate the flow of energy and momentum through the surface of the tube. With this, he assumes the existence of an infinite and negative mass in the center of the tube such that, when subtracted from the infinite and positive mass produced by the Coulomb field, which surrounds the outside of the field, results in a finite and measurable mass for the particle. In this way, the problem of infinite mass and consequently divergent self-energy, found by the Lorentz-Abraham model, was solved.

The equation of motion of a charged and accelerated particle became known as the *Lorentz-Abraham-Dirac* (LAD) equation

$$a^\mu = \frac{q}{m}F^\mu{}_\nu u^\mu + \frac{2}{3}\frac{q^2}{mc^3}\left(\delta^\mu{}_\nu + u^\mu u_\nu\right)\dot{a}^\nu. \tag{1.4}$$

Here, $F^\mu{}_\nu$ is the electromagnetic field tensor.

This model consists of an equation of motion that allows non-physical solutions. For example, a particle that is not subject to an external force can acquire an acceleration that grows exponentially with time. This solution is called a **runaway solution** and is not physical. Moreover, as discussed by Dirac in [21], when he tried to avoid the runaway solution, considering that an external force starts to act at a certain time, it causes changes in the particle's movement before the beginning of the action of that force, violating thus causality.

To avoid non-physical solutions of the dynamics of a charged particle in an electromagnetic field, Eliezer and Peierls [22] was among the first to derive equations of motion with reduced order, no more than second order. In their approach, they assume an expansion of the equation of motion containing derived terms of even higher order, with the highest-order derivative terms assumed to be small and the convergence for the equation of motion of the electron in an electric field only happens when the external frequency is much smaller than the natural frequency of the system.

In this limit, the equation of motion becomes

$$m\dot{\mathbf{v}} = e\mathbf{E} + \frac{2}{3}\frac{q^3}{mc^3}\dot{\mathbf{E}}. \tag{1.5}$$

Where $\mathbf{v}$ is the velocity of the particle, $\mathbf{E}$ is the external electric field, and the dot means a time derivative. Here, for convenience, a different notation has been used from the original work of Eliezer and Peierls. Some applications of this equation have already been made in [22].

Still, in the context of electrodynamics, a slightly different approach was obtained by Landau and Lifshitz [23] which applies the ORT as the limit where the self-force is much smaller than the external force. In this way, the self-force is treated as a perturbation. In [23], the method is applied to the non-relativistic version of the LAD equation,

$$m\dot{\mathbf{v}} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{H} + \frac{2}{3}\frac{q^2}{c^3}\ddot{\mathbf{v}}, \tag{1.6}$$

using a coordinate system in which the charge is momentarily at rest. In the above equation, $\mathbf{E}$ and $\mathbf{H}$ are the external fields and the last term is the self-force.

As mentioned above, the self-force term is considered as a perturbation. In this way, substituting the first approximation $\dot{\mathbf{v}} = q\mathbf{E}/m$ for the term of first derivative, the self-force term of the Equation (1.6) results in

$$\mathbf{f}_{self} = \frac{2}{3}\frac{q^3}{mc^3}\dot{\mathbf{E}} + \frac{2}{3}\frac{q^4}{m^2c^4}\mathbf{E}\times\mathbf{H}. \tag{1.7}$$

If there is no external field acting on the charge, $\mathbf{E} = 0$ and $\mathbf{H} = 0$, the Equation (1.6) becomes $m\dot{\mathbf{v}} = 0$ and Newton's first law is satisfied. The expression for the self-force term (1.7) was the same found by Eliezer and Peierls for the case of the electron submitted only to an electric field (1.5). The condition that the self-force is much smaller than the external force follows from the possibility of comparing the frequencies, as noted earlier by Eliezer and Peierls, that the external frequency must be much smaller than the natural frequency of the system. Thus, as Eliezer, Peierls, Landau, and Lifshitz themselves warn, this condition limits the validity of the method.

The ORT is also used in [24] and [25] in the context of self-force. Evidently, because they are perturbations on differential equations, the technique is also applied to effective gravity. To our knowledge, the first to apply the ORT in this context were the authors Bel and Zia [26], after that, also by Simon [27], [28] and Parker [29].

As far as both the point particle self-force and effective gravity are concerned, there is a drastic difference between them that we must emphasize here [1]. In the case of effective gravitational theories, it is not possible to know in advance which solutions have physical behavior. As it is well-known, solutions were found with no initial singularities [11] or with no initial singularity and with no particle horizons [30], [31], [32]. Also, instabilities as tachyon were showed by [33] and [34] for the sign of the regularization parameters $\alpha > 0$ and $\beta < 0$ in the action (1.1). The authors [11], [28], [29], [30], [31], [32] consider, for the equation of motion, an additional term with zero covariant divergence in all conformally flat models, which is due to Ginzburg [35], see also [36]. Although this term is different from the others because it does not come from the variation of the effective action, it is allowed by Wald's axioms [37], [38]. In this work, Ginzburg's term is not considered in the equation of motion. Only terms that arise by varying a consistent model of effective gravity are taken into account in a scenario where quantized fields are considered in a classical gravitational background, see, for instance, [12], [39], [40], [41], [42], and [43] the technique developed by Schwinger-DeWitt where the divergences present in the effective action can be eliminated by redefining some renormalized constants [44].

The present work aims to carefully analyse the conditions of validity of the ORT applied to Starobinsky inflation [11] and the connection of those to the slow–roll conditions.

The thesis is organized as follows:

1. Chapter 2 offers a brief review of the standard model of cosmology, the ΛCDM model;
2. Chapter 3 presents a review of inflation;
3. Chapter 4 contains the original contribution of this thesis: the analysis of the conditions of validity of the ORT applied to the case of Starobinsky inflation. The content of this chapter is based on Ref. [1].
4. The conclusions are contained in Chapter 5.

The results are obtained numerically using the *gnu/gsl ode* package[1], containing the Runge-Kutta Prince-Dormand (8, 9) method, on Linux. This method is one of those used to obtain numerical solutions to initial valued *Ordinary Differential Equations* (ODE). It converts a higher order system into a first order one set by introducing a new variable. The codes were obtained using the algebraic manipulator Maple 17.

The metric signature is $-+++$, Greek indices run over values $0-3$ and natural units $G = \hbar = c = 1$ are employed.

[1]https://www.gnu.org/software/gsl/doc/html/ode-initval.html

The standard cosmological model

The Standard Cosmological Model (ΛCDM) with cold dark matter and dark energy in the form of the cosmological constant Λ (dominating at late times) belongs to the class of the Hot Big Bang models. These are supported by the following three observational pillars:

- The CMBR is a thermal relic of a hot and high density phase of the evolution of our universe at early times. The measurement of the CMBR indicates a high degree of isotropy of the early universe. Recent observations of the large scale structure (greater than 100 Mpc) of the universe also suggests homogeneity. The cosmological principle assumes homogeneity and isotropy, i.e. at any given time, the spatial hypersurfaces are maximally symmetric. This results in the *Friedmann-Lemaître-Robertson-Walker* (FLRW) metric (2.1);

- The production of the lightest elements (principally Helium) during a primordial nucleosynthesis [45]. On the other hand, the heaviest elements in the universe were produced within the oldest stars through nuclear fusion;

- The expanding universe. Observations establish a linear relation between the recessional velocity of the galaxies and their proper distance, known as the Hubble-Lemaître's law. Mathematically, the FLRW line element can account for this law [46], [47].

There are some puzzles related to the Hot Big Bang model, such as the flatness problem and the horizon problem. Postulating a sufficiently abrupt accelerated early phase of expansion, some of these problems can be solved. This phase is called inflation and will be discussed in Chapter 3.

2.1. Friedmann-Lemaître-Robertson-Walker metric

The FLRW line element is given by:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi\right)\right], \tag{2.1}$$

where t is the cosmic time, (r, θ, φ) are the comoving (spherical) spatial coordinates, $a(t)$ is the scale factor (with the dimension of a length) and k is the constant spatial curvature of the universe, which can assume the values $+1$, -1 and 0 for spherical, hyperbolic or flat spaces of constant spatial curvature, respectively. Natural units $c = 1$ are used.

Using the well-known formula for the Levi-Civita connection:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(g_{\nu\sigma,\lambda} + g_{\lambda\sigma,\nu} - g_{\nu\lambda,\sigma}), \tag{2.2}$$

where a comma represents the partial derivative and using the FLRW metric the resulting non-zero terms of the connection are:

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2}, \Gamma^0_{22} = a\dot{a}r^2, \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta, \\
&\Gamma^1_{11} = \frac{kr}{1-kr^2}, \Gamma^1_{22} = -r\left(1-kr^2\right), \Gamma^1_{33} = -r\sin^2\theta\left(1-kr^2\right), \\
&\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}, \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \\
&\Gamma^2_{33} = -\sin\theta\cos\theta, \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.
\end{aligned} \tag{2.3}$$

Here $\dot{a} \equiv \mathrm{d}a/\mathrm{d}t$, the time dependence of the scale factor has been omitted to simplify the notation.

Using the formula for the Ricci tensor:

$$R_{\nu\beta} = R^{\mu}{}_{\nu\mu\beta} = \Gamma^{\mu}_{\nu\beta,\mu} - \Gamma^{\mu}_{\nu\mu,\beta} + \Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\mu}, \tag{2.4}$$

and the connection (2.3), the non-null terms are given by

$$\begin{aligned}
R_{00} &= -3\ddot{a}/a, & R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}, \\
R_{22} &= r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k\right), & R_{33} &= r^2\sin^2\theta\left(a\ddot{a} + 2\dot{a}^2 + 2k\right).
\end{aligned} \tag{2.5}$$

For the Ricci scalar,

$$R = R^{\nu}{}_{\nu} = g^{\nu\beta}R_{\nu\beta} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}, \tag{2.6}$$

using (2.5), one obtains:

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{2.7}$$

2.2. Friedmann equations

The Einstein-Hilbert action couples the geometry of space-time to matter as follows:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R + \int \mathrm{d}^4x\sqrt{-g}L_M, \tag{2.8}$$

where G is the gravitational constant, g is the determinant of the metric $g_{\mu\nu}$, $d^4x\sqrt{-g}$ is the invariant volume element, R is the Ricci scalar, and L_M is the Lagrangian density of the matter.

Variation of this action with respect to the metric, leads to the Einstein-Hilbert equation, as instance, see [48] and [49], $G_{\mu\nu} = \kappa T_{\mu\nu}$, with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathfrak{L})}{\delta g^{\mu\nu}}. \tag{2.9}$$

Where $G_{\mu\nu}$ is the Einstein's tensor, $T_{\mu\nu}$ the energy-momentum tensor and $\kappa = 8\pi G/c^4$ with c the speed of light.

The cosmological principle leads us to model the universe as a perfect fluid, whose energy-momentum tensor $T_{\mu\nu}$ has the following form:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.10}$$

where u^μ is the 4-velocity of a fluid element, $\rho(t)$ and $p(t)$ are the density of energy and fluid pressure, respectively. They depend only on time because of the cosmological principle, to simplify the notation this time dependence is omitted. In the comoving reference frame, we have $T^\mu{}_\nu = diag[-\rho, p, p, p]$.

The component $\mu = 0$ of the conservation of the energy-momentum tensor: $T^{\mu\nu}{}_{;\nu} = 0$, where the semi-colon represents the covariant derivative, results in

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{2.11}$$

where the FLRW metric has been employed here.

For a simple equation of state (EoS) $p = w\rho$, with w constant, the energy density evolves as $\rho \propto a^{-3(1+w)}$. Some examples of relevant w's are: $w = 1/3$ for radiation, with $\rho_r \propto a^{-4}$; $w = 0$ for cold matter, with $\rho_m \propto a^{-3}$; $w = -1$ for the vacuum energy (equivalent to the cosmological constant), with $\rho_\Lambda \propto const$.

The Einstein-Hilbert equation has the tensor of the lhs related to the geometry of the space-time and the tensor of the rhs with the matter content of the universe. This equation was initially modified by A. Einstein in 1917, he introduced a term called the cosmological constant, Λ, to counterbalance the gravitational force and keep the universe stationary, a static model of the universe at that time was expected. After Hubble found that the universe is in expansion, Einstein dropped this change in his equation. Presently, the cosmological constant has another use in the original Einstein-Hilbert equation. The general form for the actually modified equation becomes,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{2.12}$$

It follows that the cosmological constant term is equivalent to adding in the form of the energy-momentum tensor the constant energy density and isotropic pressure:

$$\rho_{vac} = \frac{\Lambda}{\kappa}, \qquad p_{vac} = -\frac{\Lambda}{\kappa}, \tag{2.13}$$

representing an equivalence of the cosmological constant to vacuum energy [50]. The negative pressure of the vacuum can accelerate the universe allowing for a desirable cosmological model.

Using the FLRW metric in the Einstein-Hilbert equations, the Friedmann equations are obtained. These are the dynamic equations that describe the evolution of the scale factor $a(t)$. Due to the isotropy of the FLRW metric, the Friedmann equations are only two:

$$\frac{\dot{a}^2}{a^2} = H^2 = \frac{\kappa}{3}\rho - \frac{k}{a^2}, \tag{2.14}$$

where the Hubble parameter is defined as:

$$H \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\dot{a}}{a}. \tag{2.15}$$

This parameter is not constant (except for the period of inflation when it is approximately constant).

Moreover,

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\kappa p. \tag{2.16}$$

If (2.14) is substituted into (2.16), one has:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6}\kappa\,(\rho + 3p). \tag{2.17}$$

Here it is considered the case where the cosmological constant is set as null $\Lambda = 0$ unless we want to include any non-zero vacuum energy density in the total energy-momentum tensor.

The first Friedmann's equation (2.14) can be reformulated as,

$$\frac{k}{(Ha)^2} = \Omega - 1, \tag{2.18}$$

where Ω is the ratio of the density to the critical density $\rho_c \equiv 3H^2/\kappa$, that is: $\Omega \equiv \rho/\rho_c$. Usually, $\Omega = \Omega_m + \Omega_r + \Omega_\Lambda$, where Ω_m refers to the matter density, Ω_r to the radiation density, and Ω_Λ to the vacuum energy density.

The term $\Omega_k = k/(Ha)^2$ is the curvature density.

2.3. Problems of the standard model

Despite its successes, the Hot Big Bang model presents some issues. Some of these will be discussed in this section. In the next chapter, it will be shown how inflation might help in solving these puzzles.

2.3.1. The horizon problem

A convenient way to write the FLRW metric is

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\chi^2\right), \tag{2.19}$$

where we chose $\theta = const$ and $\phi = const$ and thus χ is the comoving distance. The time coordinate employed here is the so-called conformal time:

$$\eta \equiv \int \frac{dt}{a(t)}. \tag{2.20}$$

Massless particles moves along null geodesics, i.e. $ds^2 = 0$, or $-d\eta^2 + d\chi^2 = 0$. Therefore, the function $\chi(\eta)$ is determined by

$$\chi(\eta) = \pm\eta + const, \tag{2.21}$$

which correspond to straight lines at angles $\pm 45°$ in the plane (η, χ), i.e. the light cone [7]. No particle travels faster than the speed of light, so causally connected events must have overlapping past light cones.

It so happens in the CMBR sky that many regions had never been in causal contact (i.e. do not have overlapping past light cones) and nonetheless share the same average temperature, see Figure 2.1.

We can elaborate a bit more on this point. If the universe has a finite age, then photons can travel only a finite distance. Thus, the causal connection is limited by a certain volume with a boundary called particle horizon, also represented in Figure 2.1.

According to (2.20) and (2.21) the comoving coordinate χ_p for the particle horizon, or the comoving horizon, is:

$$\chi_p = \eta - \eta_i = \int_{t_i}^{t} \frac{d\bar{t}}{a(\bar{t})} = \int_{a(t_i)}^{a(t)} \frac{da}{a}\frac{1}{aH}, \tag{2.22}$$

where $(aH)^{-1}$ is known as the comoving Hubble radius and η_i or t_i corresponds to the beginning of the universe.

In the Hot Big Bang model, one has that Ha diverges as $a \to 0$ (for example as $1/a$, if radiation dominates at early times). Therefore, the particle horizon also tends to zero as $a \to 0$, and events separated by a distance larger than the particle horizon are thus causally disconnected. On the other hand, it can be shown that an early phase of accelerated expansion is able to make the particle horizon arbitrarily large in the past. Indeed, note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)H(t)\right] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}a(t). \tag{2.23}$$

Thus, the comoving Hubble radius, $(aH)^{-1}$, decrease when aH increase, and in this epoch the universe is accelerates,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[a(t)H(t)\right] > 0 \Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}t^2}a(t) > 0. \tag{2.24}$$

Therefore, an early phase of accelerated expansion may provide a mechanism by which different regions in the CMBR sky were in causal contact.

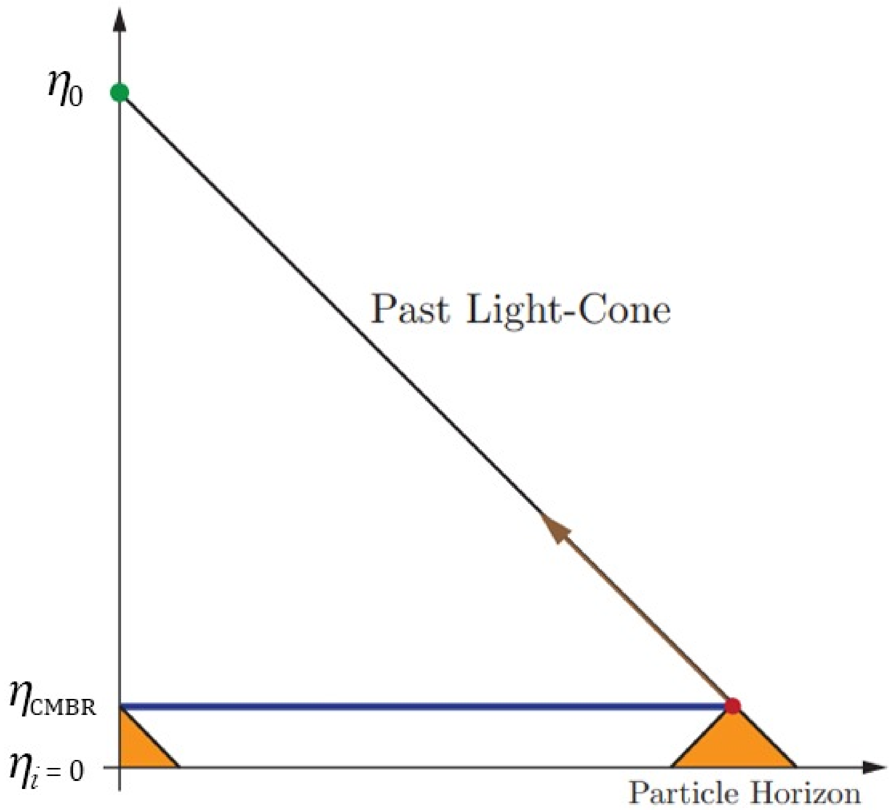

Figure 2.1.: The conformal diagram for the Hot Big Bang model, reproduced from reference [5]. The Figure shows that the CMBR sky has many regions that were never in causal contact (there is no overlap in your past light cones). It also showed the limited volume with a boundary called particle horizon for a finite age universe.

2.3.2. The flatness problem

As shown in Equation (2.18):

$$\Omega_k = \Omega - 1. \tag{2.25}$$

Here $\Omega_k = k/(Ha)^2$ is the curvature density.

If radiation dominates all the way to the Big bang, at early times we have:

$$\Omega_k \propto 1/(Ha)^2 \propto a^2 \ . \tag{2.26}$$

Therefore, the spatial curvature grows considerably.

The latest Planck data provides the following restriction for the present value of the curvature density: $\Omega_{k,0} = 0.0007 \pm 0.0019$ [3]. Therefore, we must have $\Omega_k(10^{-43}\ \mathrm{sec}) \leq O(10^{-60})$ at the Planck epoch and $\Omega_k(1\ \mathrm{sec}) \leq O(10^{-16})$ at the epoch of the primordial nucleosynthesis, in order to be consistent with observation. This means that Ω_k must be determined at the Planck scale with a fine-tuning of 60 significant digits. This fine-tuning of the initial conditions is known as the flatness problem.

Inflation

The original idea of inflation arose to explain the issues of the Hot Big Bang models mentioned in Section 2.3. Inflation is an epoch when the scale factor $a(t)$ grew quasi-exponentially, about 10^{-34} seconds after the Big Bang. Most theories model inflation via a spatially homogeneous scalar field $\phi(t)$, named inflaton, subject to a potential $V(\phi)$. Inflation occurs during a regime of the evolution of the inflaton known as slow-roll, where the field slowly moves towards the minimum of the potential. When the latter is reached, the scalar field begins to oscillate and the universe starts reheating. See Figure 3.1 for an example of inflaton potential.

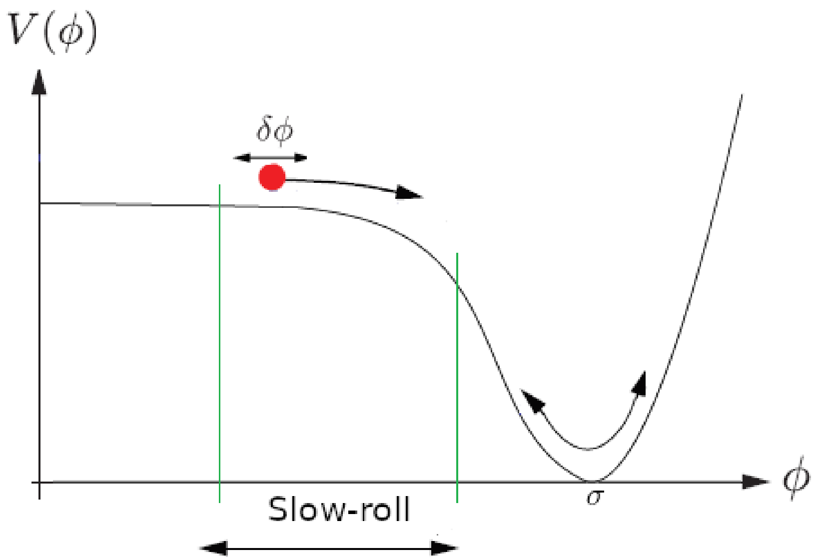

Figure 3.1.: The plot shows the regime of the evolution of the inflation when the scalar field slowly decreases to its minimum value. This regime is known as slow-roll. After that, the scalar field oscillates around the minimum $\phi = \sigma$, and the universe starts reheating. Figure reproduced from reference [5].

3.1. Definition

As we will show, an exponential expansion of the universe can be obtained if we consider a field with properties similar to a vacuum. As shown in the Section 2.2, the density of the vacuum energy is constant, $\rho_{vac} = const.$ Thus, the first Friedmann equation (2.14) results in

$$H = \sqrt{\frac{\kappa \rho_{vac}}{3}} = const. \tag{3.1}$$

The spatial curvature term was ignored in the Friedmann equation because it anticipates the huge increase in the scale factor. Thus, the term k/a^2 becomes very small and can be neglected (that is the reason why Inflation solves the flatness problem).

From the vacuum energy density given in (2.13), with $\Lambda > 0$, we have $H = \sqrt{\Lambda/3}$. Thus,

$$\frac{da}{a} = \sqrt{\frac{\Lambda}{3}} dt \Rightarrow a(t) = c \exp{(Ht)}, \tag{3.2}$$

where it is used the definition for the Hubble parameter (2.15). According to (3.1) and (3.2) the scale factor grows exponentially with time,

$$a(t) \propto \exp(Ht) \propto \exp\left(\sqrt{\frac{\Lambda}{3}} t\right). \tag{3.3}$$

As will be discussed below, inflation can solve some of the puzzles of the Hot Big Bang model. In the Subsection 2.3.1, it was shown that the particle horizon problem is a direct consequence of the deceleration of the universe. Thus, a possible solution is to consider an accelerated phase of expansion before the deceleration phase.

3.2. Dynamics of the inflaton

Neglecting interactions with matter, the inflaton Lagrangian can be written as:

$$\mathfrak{L} = -\partial^\mu \phi \partial_\mu \phi / 2 - V(\phi) = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{3.4}$$

where $\partial_i \phi = 0$ as a result of the field being homogeneous [6].

The energy-momentum tensor for the field ϕ is:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathfrak{L})}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right), \tag{3.5}$$

where the temporal and spatial terms are

$$T_{00} = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{3.6}$$

$$T_{11} = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{3.7}$$

respectively (using here the FLRW metric already).

Comparing with the definition of the energy-momentum tensor for a perfect fluid, Equation (2.10), it results in,

$$\rho = \dot{\phi}^2/2 + V(\phi), \tag{3.8}$$
$$p = \dot{\phi}^2/2 - V(\phi). \tag{3.9}$$

Thus, the EoS parameter w becomes

$$w = \frac{p}{\rho} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \tag{3.10}$$

The first Friedmann equation (2.14) for $k/a^2 \ll \rho$ is $H^2 = \kappa\rho/3$, with ρ given by (3.8) becomes $H^2 = \kappa[\dot{\phi}^2/2 + V(\phi)]/3$, deriving both sides, we obtain

$$2H\dot{H} = \frac{\kappa}{3}\left[\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right], \tag{3.11}$$

where $V'(\phi) \equiv \mathrm{d}V(\phi)/\mathrm{d}\phi$. Using the conservation of the energy-momentum tensor (2.11), and taking the derivative of the first Friedmann equation (3.1), we can get $\dot{H} = -\kappa(\rho + p)/2$ and then substituting the results from (3.8)-(3.9) it is found that

$$\dot{H} = -\frac{\kappa\dot{\phi}^2}{2}. \tag{3.12}$$

Using (3.11) and (3.12), we obtain the following equation of classic motion for the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{3.13}$$

where the second term is proportional to the Hubble factor, showing that the growth of the scalar field is damped by the expansion of the universe.

The equation of motion (3.13) can also be obtained by varying the action given by the Lagrangian (3.4), using the Euler-Lagrange formalism, or using the conservation of the energy-momentum tensor $T^{\mu\nu}{}_{;\nu} = 0$.

3.2.1. Slow-roll regime

In the slow-roll regime, the term $\ddot{\phi}$ is neglected. As a result, some conditions arise which we will discuss shortly. The equation of motion (3.13), reduces to

$$3H\dot{\phi} = -V'(\phi). \tag{3.14}$$

The first Friedmann equation (2.14) with (3.8) and $k/a^2 \ll \rho$, is given by

$$H^2 = \frac{1}{3M_{PL}^2}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right), \tag{3.15}$$

where $\kappa = 1/M_{PL}^2$ with $M_{PL} = \left(\hbar c^5/G\right)^{1/2}$ the Planck mass, c the speed of light, $\hbar$ the Planck constant and G the gravitational constant.

For a quasi-exponential expansion, it is necessary the condition $|\dot{H}|/H^2 \ll 1$, which is satisfied by

$$\dot{\phi}^2 \ll V(\phi), \tag{3.16}$$

where it is used the Equations (3.12) and (3.15). Therefore, the kinetic energy of the scalar field $\dot{\phi}^2/2$ is much smaller than its potential energy $V(\phi)$, so $\rho \simeq V(\phi)$ according to Equation (3.8), and Equation (3.15) becomes approximately

$$H^2 \simeq \frac{1}{3M_{PL}^2} V(\phi). \tag{3.17}$$

Furthermore, for the EoS parameter, Equation (3.10), the approximate value is $w \simeq -1$, i.e., the same as for the case of the energy density governed by the vacuum energy density, as mentioned in the Section 2.2. The approximately constant vacuum density implies $V(\phi) \simeq \mathit{const}$ during the slow-roll period, as can be seen in Figure 3.1.

To obtain one of the slow-roll conditions in terms of the potential we will initially calculate the derivative of the first Friedmann equation for this regime, (3.17), which becomes

$$2H\dot{H} = (V'(\phi)\dot{\phi})/(3M_{PL}^2). \tag{3.18}$$

After, using the equation of motion (3.14), it can be obtained

$$\dot{H} \simeq -\frac{1}{M_{PL}^2}\frac{\dot{\phi}^2}{2}. \tag{3.19}$$

To proceed, we divide Equation (3.18) by Equation (3.17) and replace Equations (3.14) and (3.19) in place of H and $\dot{H}$, respectively. Then, adjusting the terms and using the condition (3.16), we arrive at one of the conditions satisfied by the potential $V(\phi)$ in the slow-roll regime,

$$\left|\frac{V'(\phi)M_{PL}}{V(\phi)}\right| \ll (3)^{1/2}. \tag{3.20}$$

The rhs of the second Friedmann equation (2.17) can be written as,

$$\frac{\ddot{a}}{a} = H^2\left(1 + \frac{\dot{H}}{H^2}\right) = H^2\,(1-\epsilon), \tag{3.21}$$

where $\epsilon \equiv -\dot{H}/H^2$. Thus, according to Equation (3.21) we get, $\epsilon = 1 - \ddot{a}/aH^2$. During inflation for an accelerated expanding universe, $\ddot{a} > 0$, it is discussed before that, as a consequence, aH increases and so $\epsilon < 1$. For $a \propto e^{Ht}$ then $\epsilon = 0$. Thus, $\epsilon \ll 1$ for a quasi-exponential expansion, which is implemented by the slow-roll condition.

The slow-roll condition $\epsilon \ll 1$, can be written as $-\dot{H} \ll H^2$. Deriving both sides results in the second condition for the slow-roll period in terms of the Hubble parameter,

$$\eta \equiv -\frac{\ddot{H}}{H\dot{H}} \ll 1. \tag{3.22}$$

Both ϵ and η can be written in terms of ϕ and its derivatives when we substitute in their expressions the values of $\dot{H} = -\dot{\phi}^2/(2M_{PL}^2)$ and $\ddot{H} = -\ddot{\phi}\dot{\phi}/M_{PL}^2$ obtained when the first Friedmann equation (2.14) is substituted into the second Friedmann equation (2.17) and it is used Equations (3.8)-(3.9). Thereby, in the slow-roll approach,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2M_{PL}^2} \ll 1, \tag{3.23}$$

$$\eta \equiv -\frac{\ddot{H}}{H\dot{H}} = -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1. \tag{3.24}$$

Deriving the equation of motion (3.14) and ignoring the term that contains ϵ by the condition that this term becomes negligible during this regime $\epsilon \ll 1$, with this, we obtain,

$$\ddot{\phi} = -\frac{V''(\phi)\dot{\phi}}{3H} = \frac{V''(\phi)V'(\phi)}{9H^2}, \tag{3.25}$$

where it is substituted again the equation (3.14). The slow-roll regime allows us to state that the condition (3.16) can be see as $\ddot{\phi}/V'(\phi) \ll 1$. Thus, the Equation (3.25) results in

$$V''(\phi) \ll 9H^2, \tag{3.26}$$

substitute the Equation (3.17), we get the condition

$$\left|\frac{V''(\phi)M_{PL}^2}{V(\phi)}\right| \ll 3. \tag{3.27}$$

In the period of slow-roll, the consequence of negligence the term $\ddot{\phi}$ requires that the conditions (3.23) and (3.24) are satisfied by the Hubble parameter or by the terms of the derivatives of ϕ, and the conditions (3.20) and (3.27) are satisfied by the potential $V(\phi)$.

Dividing Equation (3.17) by Equation (3.14) to isolate H, it is possible to calculate the number of e-folds of growth in the scalar factor that occurs before the inflation end,

$$N \equiv \ln\left(\frac{a_{end}}{a_{initial}}\right) = \int_i^f H dt = \int_i^f \frac{H}{\dot{\phi}} d\phi = -\frac{1}{M_{PL}^2}\int_{\phi_i}^{\phi_f} \frac{V(\phi)}{V'(\phi)} d\phi. \tag{3.28}$$

It is enough to take $\phi_f = \sigma$, see Figure 3.1, as the minimum value of the field. That is also the end of the slow-roll period when the expansion slows down showing that inflation ends up starting the process of decay of the scalar field into other particles known as the reheating period of the universe.

For a polynomial potential, $V''(\phi) \sim V'(\phi)/\phi$, using Equation (3.17) it results in $N \sim 3H^2/V''(\phi)$ by the condition of slow-roll (3.26) we get $N \gg O(1)$ [6]. To solve the flatness and horizon problems it is necessary $N \geq 60$ [5].

3.3. Inflationary models

The conditions (3.20) and (3.27) are conveniently rewritten in terms of the dimensionless parameters ϵ_V and η_V as,

$$\epsilon_V \equiv \frac{M_{PL}^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta_V \equiv M_{PL}^2\left(\frac{V''}{V}\right), \tag{3.29}$$

where the conditions that $|\epsilon|, |\eta| \ll 1$ at the slow-roll regime implies that $\epsilon \approx \epsilon_V$ and $\eta \approx \eta_V - \epsilon_V$. Although these parameters are not observable, they are related to inflation observables, which we will see below.

Inflation is the period when quantum fluctuations may have developed primordial structures. There are two classes of such fluctuations: scalar and tensor. The scalar becomes densities of fluctuations that develop in cosmic structures, while the tensor one generates primordial gravitational waves [7], [51], [52], [53].

The amplitudes of the power spectra of the fluctuations densities, A_S, and of the gravitational wave power spectrum, A_T, expressed in terms of V and ϵ are

$$A_S(k) \simeq \frac{1}{24\pi^2}\left(\frac{V}{M_{PL}^4\epsilon}\right)\Bigg|_{k=a(\phi)H(\phi)}, \qquad A_T(k)) \simeq \frac{2}{3\pi^2}\left(\frac{V}{M_{PL}^4}\right)\Bigg|_{k=a(\phi)H(\phi)}. \tag{3.30}$$

The definition of the tensor-scalar ratio is

$$r \equiv \frac{A_T(k)}{A_S(k)} \approx 16\epsilon. \tag{3.31}$$

The scalar index is defined as

$$n_S \equiv 1 + \frac{\mathrm{d}\ln A_S(k)}{\mathrm{d}\ln k} \approx 1 + 2\eta - 6\epsilon. \tag{3.32}$$

Several theoretical inflation models can be compared with observational data through experimental constraints on r and n_S and thus analyze their validity, see Figure 3.2, reproduced from the reference [16], for a number of e-folds from 47 to 57. Among these models, we have the Starobinsky inflation, R^2, represented by the smaller orange ball, which will be discussed in the next subsection, for the number of e-folds $N = 50$. In Figure 3.2, light blue contours are a current constraint by BICEP2/Keck Array telescope taken through 2014 (BK14) [54] and by Planck mission. The red spot will generate a new type of constraint that a new mission, such as PICO, can establish [16]. If the model is above the red spot, like Starobinsky's inflationary model, it will be well detected by this future mission. Instead, if the model is below the red spot, an upper limit can be set. There is one class characterized by the monominal potential of the form $V(\phi) \propto \phi^p$, where select models are represented by blue lines. Some of the models of this class are already unfavorable by current observations. The other class is characterized by potentials that approach a constant as a function of the field value, as a power law, or exponentially. Two examples of this class are represented in Figure 3.2 by the gray and green bands. Starobinsky inflation and the Goncharov-Linde model (GL) of chaotic initial conditions in supergravity [55] (represented by the smaller purple ball), is also included in this class and predicts $r \sim 0.004$ and $r \sim 0.0004$, respectively. On the Higgs inflationary model, a scalar field is coupled non-minimally to gravity, and in [56] it is allowed to contain terms of R^2 in the action. The Higgs inflationary model is represented by the larger orange ball in Figure 3.2.

Starobinsky inflationary model is one of those that best fit the amplitude values of the tensor-to-scalar ratio, according to the latest observations of the CMBR [13], [16], [17].

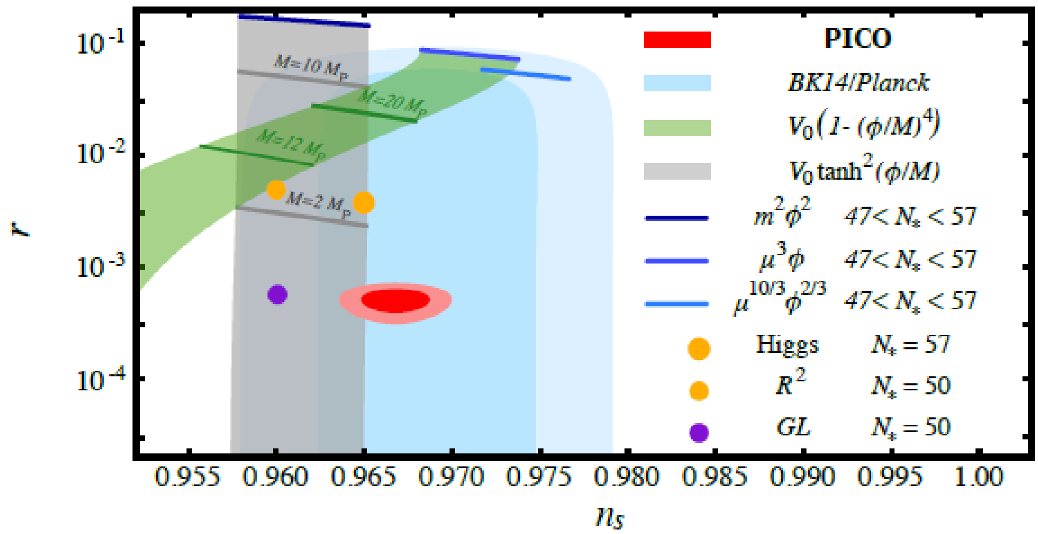

Figure 3.2.: In the figure, we see the predicted values of r and n_S for several inflationary models, for a number of e-folds from 47 to 57. The confinement contour is obtained by combining the data from a number of independent experiments. Some of these models are already excluded. Figure reproduced from reference [16].

3.3.1. Starobinsky inflationary model

The action of the gravitational $f(R)$ theory is defined as follows

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R), \tag{3.33}$$

where, as before $\kappa = 8\pi G/c^4$.

The field equations are obtained by taking the variation of the action (3.33) with respect to the metric $g_{\mu\nu}$, see the Appendix A. The equation of motion is

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' = 0, \tag{3.34}$$

where for simplicity of notation $f(R) \equiv f$ and $f' \equiv \partial f/\partial R$, the Ricci scalar dependence of $f(R)$ has been omitted to simplify the notation. The action (3.33) can be rewritten in the following way

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa}f'R - \bar{V}(\phi)\right) \tag{3.35}$$

with $\bar{V}(\phi) = (Rf' - f)/2\kappa$ and $f' \equiv \phi$, see [57].

The higher-order theory described by the action (3.35) is written in the Jordan frame. In this frame, the scalar field ϕ is non-minimally coupled with the Ricci scalar in the Lagrangian and the theory is nonlinear. It is possible to obtain this action in the Einstein frame by the conformal transformation of the metric tensor and with re-scaling of the scalar field, see Equation (3.46). In the Einstein frame, the scalar field is coupled with the matter field and the theory is linear in the Ricci scalar, similar to the general relativity theory. The conformal transformation of the metric tensor is

$$\tilde{g}_{\mu\nu} = e^{\Omega}g_{\mu\nu}, \tag{3.36}$$

$$\tilde{g}^{\mu\nu} = e^{-\Omega}g^{\mu\nu}, \tag{3.37}$$

where e^{Ω} is the conformal factor and the tilde represents the quantity in the Einstein frame. See, for instance, [7], [57], [58], and [59] for references on conformal transformation.

To obtain the relation for the Ricci scalar between the two frames in detail, see the Appendix B. The Ricci scalar in the Einstein frame written in terms of the Jordan frame is

$$\tilde{R} = e^{-\Omega}\left(R - 3\Box\Omega - \frac{3}{2}\Omega_{,\gamma}\Omega_{,}{}^{\gamma}\right). \tag{3.38}$$

Following the choice,

$$f' = e^{\Omega}, \tag{3.39}$$

with,

$$\sqrt{-\tilde{g}} = e^{2\Omega}\sqrt{-g}, \tag{3.40}$$

note that, $e^{-2\Omega}f' \to e^{-\Omega}$. Thus, substituting (3.39) and (3.40) in the action (3.35) yields

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-\tilde{g}}\left(e^{-\Omega}R - \bar{V}(\phi)e^{-2\Omega}\right). \tag{3.41}$$

After, the Ricci scalar in the Jordan frame R is isolated in (3.38) in terms of the Einstein frame and then used in (3.41), leading to the following results

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2\kappa}\left[\tilde{R} + 3e^{-\Omega}\left(\Box\Omega + 2\Omega_{,\gamma}\Omega_{,}{}^{\gamma}\right)\right] - V(\phi)\right\}, \tag{3.42}$$

where

$$V(\phi) \equiv \bar{V}(\phi)e^{-2\Omega} = \frac{\bar{V}(\phi)}{f'^2} = \frac{Rf' - f}{2\kappa f'^2}. \tag{3.43}$$

The next step is to obtain the conformal transformation in the metric for the terms inside the parentheses to the Einstein frame. Note that the transformation of the first term is not so obvious. Using (3.37) and (3.40) we get for this term,

$$\begin{aligned}\Box\Omega &= \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\Omega\right] = \frac{1}{\sqrt{-\tilde{g}}}e^{2\Omega}\partial_\mu\left[\sqrt{-\tilde{g}}e^{-\Omega}\tilde{g}^{\mu\nu}\partial_\nu\Omega\right]\\ &= \frac{1}{\sqrt{-\tilde{g}}}e^{2\Omega}\left[-\sqrt{-\tilde{g}}e^{-\Omega}\tilde{g}^{\mu\nu}\partial_\nu\Omega\partial_\mu\Omega + \sqrt{-\tilde{g}}e^{-\Omega}\tilde{\Box}\Omega\right]\\ &= e^{\Omega}\tilde{\Box}\Omega - e^{\Omega}\tilde{g}^{\mu\nu}\partial_\nu\Omega\partial_\mu\Omega. \end{aligned} \tag{3.44}$$

Therefore, the conformal transformation (3.37) and the resulting transformation to the $\Box\Omega$, (3.44), is used in (3.42) to become,

$$S = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{1}{2\kappa}\left[\tilde{R} - \frac{3}{2}\tilde{g}^{\mu\nu}\partial_\nu\Omega\partial_\mu\Omega\right] - V(\phi)\right\}. \tag{3.45}$$

For the conformal transformation (3.36) in the Einstein frame, it is defined according to [60] as

$$\tilde{\phi} = \sqrt{\frac{3}{2\kappa}}\ln\phi, \tag{3.46}$$

to obtain the Einstein-Hilbert equations with a scalar field coupled with the matter, a potential $V(\tilde{\phi})$ given by a re-scaling in (3.43) and the energy-momentum tensor given by (3.5), see for instance, [57], where $e^{\Omega} \equiv \phi$. Thus, the action (3.45) becomes,

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\partial_\mu\tilde{\phi} - V(\tilde{\phi})\right]. \tag{3.47}$$

Defining the energy-momentum tensor (3.5) and the Lagrangian (3.4), we see that this action leads to the Einstein-Hilbert equation $G_{\mu\nu} = T_{\mu\nu}(\tilde{\phi})$ and to the equation of classic motion for the scalar field (3.13), according to the choice and definition of (3.39) and (3.46), see [60]. Therefore, the gravitational $f(R)$ theory is conformally equivalent to Einstein-Hilbert gravity with an additional scalar field coupled with the matter. In Appendix C it is shown for the inflationary Starobinsky model $f(R) = R + \beta R^2$, with the FLRW metric employed, that these equations lead to the higher order derivative field equations in the Jordan frame.

If the scalar field satisfies the conditions of the slow-roll period, then we have an inflationary solution in the Einstein conformal frame. However, we also have an inflationary solution in Jordan's original frame [7].

For a given model like the Starobinsky inflation the potential $\bar{V}(\phi)$ seen in (3.35) becomes

$$\bar{V}(\phi) = \frac{1}{2\kappa}(f'R - f) = \frac{1}{2\kappa}[R(1+2\beta R) - R - \beta R^2] = \frac{1}{2\kappa}(\beta R^2) = \frac{1}{2\kappa}\frac{1}{4\beta}(1-f')^2. \tag{3.48}$$

Using the choice (3.39) and the definition made before for $e^{\Omega} \equiv \phi$, we have $f' = \phi$. Thus,

$$\bar{V}(\phi) = \frac{1}{2\kappa}\frac{1}{4\beta}(1-\phi)^2 = M_{PL}^2 \frac{1}{8\beta}(1-\phi)^2, \tag{3.49}$$

in terms of the Planck mass M_{PL}, see Figure 3.3. The inflationary solution of Starobinsky in Jordan's frame was previously obtained by Ruzmaikina and Ruzmaikin [61].

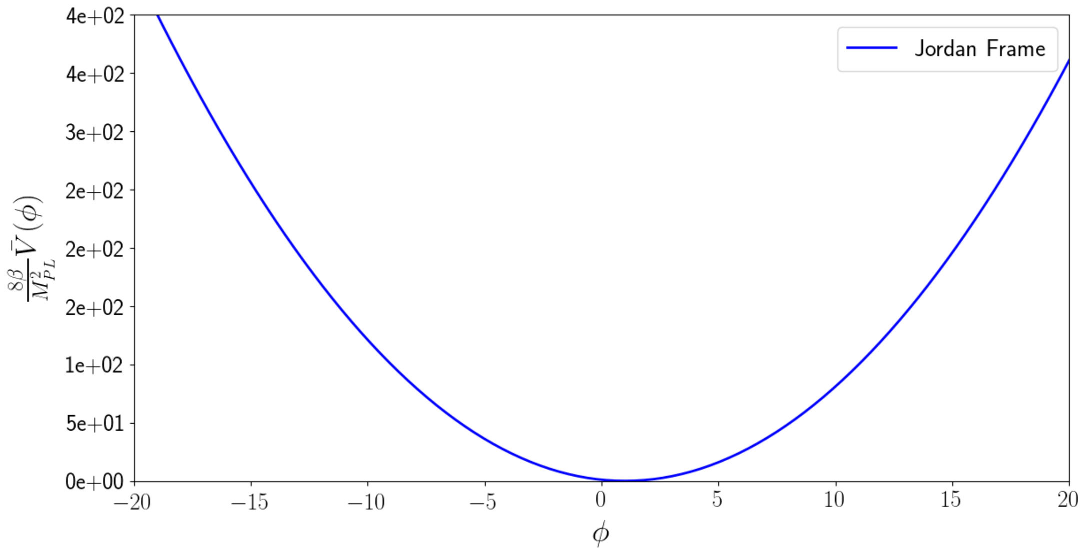

Figure 3.3.: In this figure, the graph of the Starobinsky potential × scalar field on the Jordan frame for the Equation (3.49) is plotted in blue, where the regularization parameter β must be positive to avoid the tachyon.

According to the definition of (3.46),

$$e^{\Omega} = f' = \exp\left(\sqrt{\frac{2\kappa}{3}}\tilde{\phi}\right), \tag{3.50}$$

this yields the following potential in the Einstein frame for Starobinsky inflation [62], according to Equation (3.43),

$$V(\tilde{\phi}) = \frac{1}{2\kappa}\frac{1}{4\beta}\left[1-\exp\left(-\sqrt{\frac{2\kappa}{3}}\tilde{\phi}\right)\right]^2 = M_{PL}^2\frac{1}{8\beta}\left[1-\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\right]^2, \tag{3.51}$$

see Figure 3.4, in terms of the Planck mass M_{PL}.

In the limit where $\tilde{\phi} \to \infty$ the Starobinsky potential tends to a constant value in the Einstein frame, leading to a slow-roll regime for $\tilde{\phi} \gg M_{PL}$ [58].

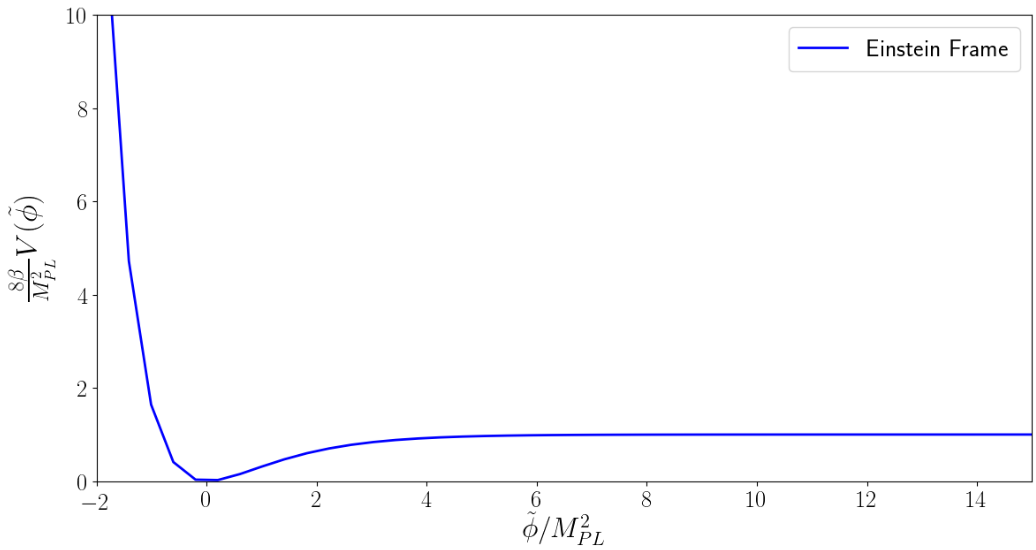

Figure 3.4.: In this figure, the graph of the Starobinsky potential × scalar field on the Einstein frame is plotted in blue for the Equation (3.51), where the regularization parameter β must be positive to avoid the tachyon.

Calculating the derivative of the Starobinsky potential in the Einstein frame (3.51) with respect to the scalar field $\tilde{\phi}$ yields,

$$V'(\tilde{\phi}) = M_{PL}^2 \frac{1}{8\beta} \frac{2}{M_{PL}} \sqrt{\frac{2}{3}} \exp\left(-\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_{PL}}\right) \left[1 - \exp\left(-\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_{PL}}\right)\right], \tag{3.52}$$

where $V'(\tilde{\phi}) \equiv \mathrm{d}V(\tilde{\phi})/\mathrm{d}\tilde{\phi}$.

To get the number of e-folds, substitute Equation (3.51) and its derivative (3.52) into the expression (3.28), which gives the following,

$$\begin{aligned} N &= -\frac{8\pi}{M_{PL}^2} \int_{\tilde{\phi}_i}^{\tilde{\phi}_f} \frac{V(\tilde{\phi})}{V'(\tilde{\phi})} d\tilde{\phi} = -\frac{8\pi}{M_{PL}} \frac{1}{2} \int \left[\sqrt{\frac{3}{2}} \frac{\left[1 - \exp\left(-\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_{PL}}\right)\right]}{\exp\left(-\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_{PL}}\right)} \right] d\tilde{\phi} \\ &= -\frac{8\pi}{M_{PL}} \frac{1}{2} \int \sqrt{\frac{3}{2}} \left[\exp\left(\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_{PL}}\right) - 1 \right] d\tilde{\phi} \approx \frac{3}{4} \exp\left(\sqrt{\frac{2}{3}} \frac{\tilde{\phi}}{M_{PL}}\right). \end{aligned} \tag{3.53}$$

Substituting Equation (3.51) and its derivative (3.52) into the parameter (3.29), we get,

$$\epsilon_V = \frac{M_{PL}^2}{2}\left(\frac{V'(\tilde{\phi})}{V(\tilde{\phi})}\right)^2 = \frac{4}{3}\frac{\exp\left(-2\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)}{\left[1-\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\right]^2} = \frac{4}{3}\left[1-\exp\left(\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\right]^{-2}. \tag{3.54}$$

In terms of the number of e-folds (3.53) the parameter ϵ_V (3.54) can be rewritten as,

$$\epsilon_V \approx \frac{3}{4N^2}. \tag{3.55}$$

By deriving Equation (3.52) with respect to the scalar field $\tilde{\phi}$ we obtain the second derivation of the Starobinsky potential in the Einstein frame

$$V''(\tilde{\phi}) = M_{PL}^2\frac{1}{8\beta}\left\{\frac{4}{3}\frac{1}{M_{PL}^2}\exp\left(-2\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right) - \frac{4}{3}\frac{1}{M_{PL}^2}\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\left[1-\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\right]\right\} \tag{3.56}$$

where $V''(\tilde{\phi}) \equiv \mathrm{d}V'(\tilde{\phi})/\mathrm{d}\tilde{\phi}$. The use of Equations (3.51) and (3.56) in the second parameter for the slow-roll period (3.29), leads to

$$\eta_V = M_{PL}^2\left(\frac{V''(\tilde{\phi})}{V(\tilde{\phi})}\right) = \frac{4}{3}\frac{\exp\left(-2\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)}{\left[1-\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\right]^2} - \frac{4}{3}\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\left[1-\exp\left(-\sqrt{\frac{2}{3}}\frac{\tilde{\phi}}{M_{PL}}\right)\right]^{-1}. \tag{3.57}$$

Therefore, using Equation (3.53) for the number of e-folds,

$$\eta_V \approx -\frac{1}{N} + O\left(\frac{1}{N^2}\right). \tag{3.58}$$

One can find the tensor-to-scalar ratio (3.31) and the scalar index (3.32) in terms of the parameters (3.55) and (3.58), where, as mentioned above in the Section 3.3, the conditions $|\epsilon|, |\eta| \ll 1$ in the slow-roll regime lead to $\epsilon \approx \epsilon_V$ e $\eta \approx \eta_V - \epsilon_V$,

$$r \approx \frac{12}{N^2}, \qquad n_S \approx 1 - \frac{2}{N} + O\left(\frac{1}{N^2}\right). \tag{3.59}$$

For $N = 50$, $r = 0.0048$ and $n_S = 0.96$. These values for the Starobisnky inflationary model are consistent with the Figure 3.2 represented by the smaller orange ball.

CHAPTER 4.

The order reduction technique

In the study of the dynamics of a charged particle, it is well known that some equations with higher order derivative can present non-physical solutions, such as runaway solutions or problems with causality [21]. Instabilities in some of the higher order derivative theories have been known since Ostrogradsky's time [63]. A dynamical variable that suffers from Ostrogradsky's instability will carry kinetic terms with opposite signs, which through couplings can result in infinite energy transfers between degrees of freedom, while the total energy of the system is conserved. This type of instability is known today as ghosts [64], [65], [66], [67]. It should be noted that it is not possible to eliminate the ghosts by a suitable choice of theory parameters, as is the case with the tachyon mentioned in the introduction in the action (1.1).

On the other hand, as known in the literature, it is possible to eliminate the ghosts in the gravitational $f(R)$ theory [66], [68]. In this case, it violates the assumption for Ostrogradsky's instability, the nondegeneracy [63]. As mentioned in [66], the field equations from the gravitational $f(R)$ theory results in only a single higher order derivative equation that carry the dynamics. The lower order derivative equation results in a constraint that limit the lower derivative degrees of freedom. This result can be shown by the conformal equivalence between the gravitational $f(R)$ theory in the Jordan frame with the Einstein-Hilbert gravity in the Einstein frame, as mentioned early in Subsection 3.3.1. In Appendix C, the argument is verified through this conformal equivalence between the two frames for the equation of motion of the Starobinsky inflation, with the FLRW metric applied.

In order to avoid non-physical solutions in the context of electrodynamics, Landau and Lifshitz were the among the first to apply the ORT [23], precisely in connection with perturbative treatments of self-force. Evidently, because they are perturbations on differential equations, the authors Simon and Parker [29] apply the ORT to quadratic gravity equations and obtain reduced equations awaited to contain fewer instabilities than the original higher order equations, this way it is easier to identify and remove non-physical solutions.

This chapter reviews the literature on the Ostrogradsky theorem and the LAD equation. This is followed by a brief presentation of the reduction method used by Landau and Lifshitz [23] and the exact solutions to the relativistic LAD equation for observers with constant proper acceleration and constant external force, found by Dirac [21], see also [19]. To emphasize the ORT, the perturbative method is applied for the relativistic LAD equation without external gravitational fields and a constant source. Also, the particular case of a harmonic oscillator is tested shortly thereafter [69]. And finally, as a more realistic case, the application and analysis of the validity conditions of the ORT applied to Starobinsky inflation and the connection of those to the slow-roll conditions is verified [1].

4.1. Ostrogradsky instability

Ostrogradsky's theorem associates a linear instability in the Hamiltonian resulting from nondegenerate Lagrangian that depends on time derivative greater than first order [63]. As a result, the energy of the system is unbounded below or above [68]. This is a direct consequence of the fact that the Hamiltonian is a linear function of increasing momentum instability as the order of the time derivative associated with the Lagrangian increases [66], [68].

We suppose here for simplicity a point particle in one dimension whose position as a function of time is described by $q(t)$.

To begin with, the well known case in the literature in which the equations of motion are of second-order will be shown. In this particular case, the Hamiltonian does not show signs of instabilities. The Lagrangian is of the form $L = L(q, \dot{q})$ with $q = q(t)$ and $\dot{q} = \mathrm{d}q/\mathrm{d}t$, the time dependence of the particle position has been omitted to simplify the notation. The Euler-Lagrange equation is written as [70], [71], [72],

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = 0. \tag{4.1}$$

As mentioned, the Lagrangian is assumed to be nondegenerate, which means that $\partial L/\partial \dot{q}$ depends on $\dot{q}$. And under these conditions, the Euler-Lagrange equation can be put in the following Newtonian form

$$\ddot{q} = F(q, \dot{q}) \Rightarrow q(t) = \Upsilon(t, q_0, \dot{q}_0). \tag{4.2}$$

The solution depends on the initial conditions $q_0 = q(0)$ and $\dot{q}_0 = \dot{q}(0)$.

The solution needs two initial conditions, which means it must have two canonical coordinates, Q, and P. It is common to find them defined in the literature as

$$Q \equiv q, \qquad\qquad P \equiv \frac{\partial L}{\partial \dot{q}}. \tag{4.3}$$

The nondegeneracy assumption makes it possible to invert the phase space transformation (4.3) to solve $\dot{q}$ in terms of Q and P. That is, there is a function $v(Q, P)$ such that,

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{q=Q,\dot{q}=v} = P. \tag{4.4}$$

The Legendre transform on $\dot{q}$ results in the canonical Hamiltonian,

$$H(Q, P) \equiv P\dot{q} - L = Pv(Q, P) - L(Q, v(Q, P)). \tag{4.5}$$

It can be seen that the evolution of the canonical equations reproduce Equations (4.1) and (4.3)

$$\dot{Q} \equiv \frac{\partial H}{\partial P} = v + P\frac{\partial v}{\partial P} - \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = v, \tag{4.6}$$

$$\dot{P} \equiv -\frac{\partial H}{\partial Q} = -P\frac{\partial v}{\partial Q} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}}\frac{\partial v}{\partial P} = \frac{\partial L}{\partial q}, \tag{4.7}$$

where it is used the phase space transformation (4.4).

When the Lagrangian is associated with first order temporal derivative, the Hamiltonian is associated with conserved quantities and the equation of motion results in second order equations in time. See [70], [71], [72].

The Ostrogradsky instability [63] arise in Lagrangians that depend on time derivative greater than first order [66], [68]. Setting up a system where the Lagrangian is $L = L(q, \dot{q}, \ddot{q})$ and nondegenerate in $\ddot{q}$. The Euler-Lagrange equation for this case is written as,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} = 0. \tag{4.8}$$

As before, q is a function of time and dot represents the derivative with respect to time.

The Equation (4.8) can be written very differently from the Newton form discussed earlier,

$$\ddddot{q} = F(q, \dot{q}, \ddot{q}, \dddot{q}) \Rightarrow q(t) = \Upsilon(t, q_0, \dot{q}_0, \ddot{q}_0, \dddot{q}_0). \tag{4.9}$$

In this case, the solution needs four initial conditions. This implies four canonical coordinates, so in Ostrogradsky choose [63],

$$Q_1 \equiv q, \qquad P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}, \tag{4.10}$$

$$Q_2 \equiv \dot{q}, \qquad P_2 \equiv \frac{\partial L}{\partial \ddot{q}}. \tag{4.11}$$

As in the previous case, the nondegeneracy assumption makes it possible to invert the phase space transformation (4.10)-(4.11) to solve $\ddot{q}$ in terms of Q_1, Q_2, and P_2. Which means that there is a function $a(Q_1, Q_2, P_2)$ such that,

$$\left.\frac{\partial L}{\partial \ddot{q}}\right|_{q=Q_1, \dot{q}=Q_2, \ddot{q}=a} = P_2. \tag{4.12}$$

The Ostrogradsky Hamiltonian can be obtained by the same method as before, through the Legendre transform on $\dot{q} = q^1$ and $\ddot{q} = q^2$,

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &\equiv \sum_{i=1}^{2} P_i q^{(i)} - L \\ &= P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2)), \end{aligned} \tag{4.13}$$

where $\mathrm{d}H/\mathrm{d}t = 0$ and H can be called the conserved energy of the system.

As before, the evolutions of the canonical equations can be verified,

$$\dot{Q}_i \equiv \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i \equiv -\frac{\partial H}{\partial Q_i}. \tag{4.14}$$

For $\dot{Q}_1$,

$$\dot{Q}_1 \equiv \frac{\partial H}{\partial P_1} = Q_2 = \dot{q}, \tag{4.15}$$

according (4.11).

For $\dot{Q}_2$,

$$\dot{Q}_2 \equiv \frac{\partial H}{\partial P_2} = a + P_2\frac{\partial a}{\partial P_2} - \frac{\partial L}{\partial \ddot{q}}\frac{\partial a}{\partial P_2} = a, \tag{4.16}$$

in which it is used the phase space transformation (4.12).

For $\dot{P}_1$,

$$\dot{P}_1 \equiv -\frac{\partial H}{\partial Q_1} = -P_2\frac{\partial a}{\partial Q_1} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \ddot{q}}\frac{\partial a}{\partial Q_1} = \frac{\partial L}{\partial q}, \tag{4.17}$$

where it is used the phase space transformation (4.12) and reproduces the Euler-Lagrange equation (4.8) together with (4.10).

For $\dot{P}_2$,

$$\dot{P}_2 \equiv -\frac{\partial H}{\partial Q_2} = -P_1 - P_2\frac{\partial a}{\partial Q_2} + \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \ddot{q}}\frac{\partial a}{\partial Q_2} = -P_1 + \frac{\partial L}{\partial \dot{q}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}, \tag{4.18}$$

where the phase space transformation (4.12) is used together with (4.10) and the Euler-Lagrange equation is reproduced (4.8).

Therefore, Ostrogradsky's choice generates the temporal evolution of the system. However, as can be seen in the Ostrogradsky Hamiltonian on (4.13), there is a linear dependence of the momentum P_1. As will be discussed below, this dependence implies that the energy is unbounded below or above [68].

The Hamiltonian (4.13) is third-order in q because it is linear in P_1 which the following format

$$H = -\dot{q}\dddot{q} + \textit{lower order terms}, \tag{4.19}$$

see, for instance, [68]. By setting the initial values q, $\dot{q} \neq 0$ and $\ddot{q}$, $\dddot{q}$ can be freely chosen and then H become unbounded.

The Hamiltonian in this case is unbounded below or above. By the initial assumption that energy is unbounded from below and bounded from above. Then, we changed the signs of L and H and notice that the energy now is bounded from below and unbounded from above. This is not surprising because in this case there is no preference for the sign of the Hamiltonian [68].

If higher order derivatives are added to the Lagrangian, the instability in the Hamiltonian will become larger [66]. In this case, the Lagrangian is of the form $L = L(q, \dot{q}, ..., q^{(N)})$ and assumed to be nondegenerate in $q^{(N)}$. The Euler-Lagrange equation becomes

$$\sum_{i=0}^{N}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \frac{\partial L}{\partial q^i} = 0, \tag{4.20}$$

The choices for the canonical Ostrogradsky coordinates are

$$Q_i \equiv q^{(i-1)}, \qquad P_i \equiv \sum_{j=i}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j-i} \frac{\partial L}{\partial q^{(j)}}. \tag{4.21}$$

As before, the nondegeneracy condition allows to invert the phase space transformation (4.21) and write $q^{(N)}$ in terms of P_N and Q_i's. Thus, there exists a function $V(Q_1, ..., Q_N, P_N)$, such that,

$$\left.\frac{\partial L}{\partial q^{(N)}}\right|_{q^{(i-1)}=Q_i, q^{(N)}=V} = P_N. \tag{4.22}$$

The Ostrogradsky Hamiltonian is obtained through the Legendre transform on $q^{(N)} = q^{(i)}$,

$$\begin{aligned} H(Q_1, Q_2, ..., Q_N, P_1, P_2, ..., P_N) &\equiv \sum_{i=1}^{N} P_i q^{(i)} - L \\ &= P_1 Q_2 + P_2 Q_3 + ... + P_{N-1} Q_N + P_N V(Q_1, ..., Q_N, P_N) - L(Q_1, ..., Q_N, V), \end{aligned} \tag{4.23}$$

and by the following evolution equations

$$\dot{Q}_i \equiv \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i \equiv -\frac{\partial H}{\partial Q_i}, \tag{4.24}$$

the reproduction of the canonical transformations (4.21) and the Euler-Lagrange equation (4.20) can be verified.

The Hamiltonian becomes linear at moments P_1, P_2, ..., P_{N-1} and therefore, for the same reason as above, is bounded from above or below. Note that only P_N can be bounded.

4.2. Particular situations

4.2.1. Lorentz-Abraham-Dirac equation

The classical theory of a charged and accelerated particle was conceived by H. A. Lorentz at the beginning of the last century. Coming from the work of Sir J. Larmor, who equated the energy loss radiated by a harmonic oscillator with a damping force, Lorentz was able to consider a very simple model in which the electrons could be bound elastically and the system treated as a charged harmonic oscillator. The electron's motion needs to change to ensure that the sum of mechanical energy and energy radiated to infinity remains constant. This energy balance manifests itself as a force acting back on the particle, called electromagnetic self-force [19], [73], [74]. The self-force term appears in the Lorentz equation of motion and is proportional to the third derivative of the particle position, as seen before in the equation (1.2),

$$\mathbf{f}_{\mathbf{self}} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{a}}. \tag{4.25}$$

Here q and m are the charge and mass of the particle, respectively, and c is the speed of light.

After that, Abraham considered a rigid sphere with spherically symmetric charge distribution as a model for the electron [20]. However, in 1904, Abraham realized inconsistency of his model with the LT. Right after these transformations were better established by Einstein, Abraham [18] abandoned his rigid electron by an electron whose shape change under LT and gives a relativistic generalization for the self-force term found by Lorentz (4.25), as mentioned in the introduction, Equation (1.3),

$$\frac{2}{3}\frac{q^2}{c^3}\left(g^{\mu}{}_{\nu} + u^{\mu}u_{\nu}\right)\dot{a}^{\nu}. \tag{4.26}$$

Where $g_{\mu\nu}$ is the metric tensor, $u^{\mu} = \mathrm{d}z/\mathrm{d}\tau$ the four-velocity, $a^{\mu} = \mathrm{d}u^{\mu}/\mathrm{d}\tau$ the four-acceleration and $\dot{a}^{\mu} = \mathrm{d}a^{\mu}/\mathrm{d}\tau$ the first derivative of the four-acceleration of the particle along the world line which is described by the relation $z(\tau)$, where τ represents the proper-time.

Nevertheless, the Lorentz-Abraham model does not hold in the case where the electron radius tends to zero, which results in a divergent self-energy. Dirac solves this problem by generalizing the Lorentz-Abraham model to a point model of the electron [21]. Dirac showed that the field responsible for the radiation can be better written as a combination of the retarded and advanced electromagnetic fields. Also, he considered a thin tube surrounding the electron world line in space-time to calculate the flow of energy and momentum through the surface of the tube. With this, he assumes the existence of an infinite and negative mass in the center of the tube such that, when subtracted from the infinite and positive mass produced by the Coulomb field, which surrounds the outside of the field, results in a finite and measurable mass for the particle. In this way, the problem of infinite mass and consequently divergent self-energy, found by the Lorentz-Abraham model, was solved.

As enunciated in the introduction, Equation (1.4), the equation of motion of a charged and accelerated particle became known as the *Lorentz-Abraham-Dirac* (LAD) equation

$$a^{\mu} = \frac{q}{m}F^{\mu}{}_{\nu}u^{\mu} + \frac{2}{3}\frac{q^2}{mc^3}\left(\delta^{\mu}{}_{\nu} + u^{\mu}u_{\nu}\right)\dot{a}^{\nu}, \tag{4.27}$$

in absence of external gravitational fields. Here $F^{\mu}{}_{\nu}$ is the electromagnetic field tensor. The first term of the LAD equation, (4.27), refers to the external force acting on the particle and the second to the self-force.

However, this model results in an equation of motion that allows non-physical solutions. For example, a particle that is not subjected to an external force can acquire an acceleration that grows exponentially with time. This solution is called a runaway solution and is not observed in nature. Moreover, as discussed by Dirac in [21], when he tried to avoid the runaway solution, considering that an external force starts to act at a certain time, it causes changes in the particle's movement before the beginning of the action of that force. This is not expected, as we know that the equation of motion certainly needs to be causal.

The LAD equation for a curved space-time can be seen in [12] and [24].

4.2.1.1. Problems of the Lorentz-Abraham-Dirac equation

As an example of the emergence of non-physical solutions associated with the Lorentz-Abraham-Dirac equation, let us consider the simplest case shown in [73]. It is considered a non-relativistic limit of a charged particle accelerated by a constant external force, $\mathbf{F}$, which acts only during the time interval $0 \leq t \leq T$. The LAD equation, (4.27), reduces to

$$\mathbf{a}(t) = \frac{\mathbf{F}}{m} + \frac{2}{3}\frac{q^2}{mc^3}\dot{\mathbf{a}}(t), \tag{4.28}$$

with solution,

$$\mathbf{a}(t) = \begin{cases} [(\mathbf{F}/m) + \mathbf{C}]e^{t/\tau} & if \quad t \leq 0, \\ [(\mathbf{F}/m) + \mathbf{C}e^{t/\tau}] & if \quad 0 \leq t \leq T, \\ [(\mathbf{F}/m)e^{-T/\tau} + \mathbf{C}]e^{t/\tau} & if \quad t \geq T. \end{cases} \tag{4.29}$$

Here the continuity condition at $t = 0$ and $t = T$ has been imposed, $\mathbf{C}$ is a constant and $\tau \equiv 2q^2/3mc^3$.

This solution shows that, if an external force acts at a certain moment, it causes changes in the particle's movement before the beginning of the action of that force, violating thus causality. Also, independently of whether the charged particle is subjected to an external force or not, the particle can gain an acceleration that grows exponentially with time. This solution is called a runaway solution and is not observed in nature. It can be shown that it is not possible to eliminate runaway and pre-acceleration problems with the same choice of integration constant.

4.2.1.2. Landau and Lifshitz method

In order to avoid non-physical solutions of the dynamics of a charged particle in an electromagnetic field, Eliezer and Peierls [22] was among the first to derive equations of motion with reduced order, no more than second order. In their approach, they assume an expansion of the equation of motion containing derived terms of even higher order, with the highest-order derivative terms assumed to be small and the convergence for the equation of motion of the electron in an electric field only happens when the external frequency is much smaller than the natural frequency of the system. In this limit, the equation of motion becomes Equation (1.5), as mentioned early in the introduction. The authors themselves made some applications of this equation [22].

Still, in the context of electrodynamics, a slightly different approach was obtained by Landau and Lifshitz [23] which applies the ORT as the limit where the self-force is much smaller than the external force. In this way, the self-force is treated as a perturbation. In [23], the method is applied to the non-relativistic version of the LAD equation, seen before Equation (1.6),

$$m\dot{\mathbf{v}} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{H} + \frac{2}{3}\frac{q^2}{c^3}\ddot{\mathbf{v}}, \tag{4.30}$$

using the reference system that the charge is momentarily at rest. Here, $\mathbf{E}$ and $\mathbf{H}$ are the external fields and the last term is the electromagnetic self-force.

The self-force term is considered as a perturbation and the second derivative of velocity can be written as

$$\ddot{\mathbf{v}} = \frac{q}{m}\dot{\mathbf{E}} + \frac{q}{mc}\dot{\mathbf{v}} \times \mathbf{H}. \tag{4.31}$$

Then, substituting the first approximation $\dot{\mathbf{v}} = q\mathbf{E}/m$ for the term of the first derivative, the self-force term of Equation (4.30) results in

$$\mathbf{f}_{self} = \frac{2}{3}\frac{q^3}{mc^3}\dot{\mathbf{E}} + \frac{2}{3}\frac{q^4}{m^2c^4}\mathbf{E} \times \mathbf{H}, \tag{4.32}$$

as mentioned before in the introduction, Equation (1.7).

The expression for the self-force (4.32) was the same found by Eliezer and Peierls for the case of the electron submitted only to an electric field (1.5).

Through the condition that the self-force is much smaller than the external force, $\mathbf{f}_{self} \ll e\mathbf{E}$, and let ω the frequency of motion, we have $\dot{\mathbf{E}} \propto \omega\mathbf{E}$, so from the terms of the self-force (4.32), the following conditions are obtained;

$$\frac{q^2}{mc^3}\omega \ll 1, \qquad\qquad \mathbf{H} \ll \frac{m^2c^4}{q^3}. \tag{4.33}$$

Introducing the wavelength of the incident radiation $\lambda \sim c/\omega$, we have from the first condition (4.33),

$$\lambda \gg \frac{q^2}{mc^2}. \tag{4.34}$$

Here q^2/mc^2 is the electric charge "radius". In this case, the wavelength of the incident radiation is large when compared to the "radius" of the electric charge q^2/mc^2, or on the other hand, the frequency of movement is small when subjected to the same comparison. Furthermore, by the second condition, it is also necessary that the external magnetic field, $\mathbf{H}$, not be too large.

The condition that the self-force is much smaller than the external force, $\mathbf{f}_{self} \ll e\mathbf{E}$ follows from the possibility of comparing the frequencies, as noted earlier by Eliezer and Peierls, that the external frequency must be much smaller than the natural frequency of the system. Thus, as Eliezer, Peierls, Landau, and Lifshitz themselves warn, this condition limits the validity of the method.

4.2.1.3. Exact solution

Following [19] and [21], observers with constant proper acceleration constitute exact solutions for the LAD equation.

In the context of the charged particle, the relativistic LAD equation in the absence of the presence of gravitational fields becomes, Equation (4.27),

$$a^\mu = \frac{q}{m}F^\mu{}_\nu u^\nu + \frac{2}{3}\frac{q^2}{mc^3}\left(g^\mu{}_\nu + u^\mu u_\nu\right)\dot{a}^\nu. \tag{4.35}$$

The second term on the right is the self-force term.

On the other hand, the Rindler coordinates (ξ, τ) are

$$x^\mu = (\xi \sinh \tau, \xi \cosh \tau, y, z), \tag{4.36}$$

where τ is the proper time and ξ the inverse of the modulus of the proper acceleration of the particle [75]. Orbits with $\xi = const.$ describe particles with constant proper acceleration and are exact solutions of the LAD equation (4.35), when the external force, $F_{\mu\nu}$, is constant [19], [21]. In this case, the self-force term is identically null.

In terms of the Cartesian coordinates the components in the Rindler coordinate (4.36) are;

$$\begin{aligned} t &= \xi \sinh \tau, & y &= y, \\ x &= \xi \cosh \tau, & z &= z. \end{aligned} \tag{4.37}$$

By taking the derivatives in (4.37),

$$\begin{aligned} dt &= \xi \cosh \tau d\tau + \sinh \tau d\xi, & dy &= dy, \\ dx &= \xi \sinh \tau d\tau + \cosh \tau d\xi, & dz &= dz, \end{aligned} \tag{4.38}$$

the line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2, \tag{4.39}$$

becomes,

$$ds^2 = -\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2. \tag{4.40}$$

Rewriting the line element (4.40) as,

$$ds^2 = \left(-\xi^2 + \frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} \right) d\tau^2, \tag{4.41}$$

for ξ, y and z constant, we get $ds^2 = -\xi^2 d\tau^2$ and thus,

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{1}{\xi}. \tag{4.42}$$

Through the line element (4.40) and the relation (4.42) we can calculate the derivatives of a particle positioned at $\xi = const.$ in Rindler coordinates,

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{1}{\xi} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{4.43}$$

Thus,

$$u^\mu = (\cosh \tau, \sinh \tau, 0, 0), \tag{4.44}$$

$$a^\mu = \frac{1}{\xi} (\sinh \tau, \cosh \tau, 0, 0), \tag{4.45}$$

$$\dot{a}^\mu = \frac{1}{\xi^2} (\cosh \tau, \sinh \tau, 0, 0). \tag{4.46}$$

In the particular case where the particle is subject only to an electric field in the $x-$direction, the electromagnetic field tensor becomes

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.47}$$

Substituting the Equations (4.44)-(4.47) into the LAD equation, Equation (4.35), results in

$$\frac{1}{\xi}\left(\sinh\tau, \cosh\tau, 0, 0\right) = \frac{q}{m}\begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cosh\tau \\ \sinh\tau \\ 0 \\ 0 \end{pmatrix} = \frac{q}{m}E\left(\sinh\tau, \cosh\tau, 0, 0\right), \tag{4.48}$$

where the self-force term is canceled in the Rindler coordinates;

$$(g^{\mu}{}_{\nu} + u^{\mu}u_{\nu})\dot{a}^{\nu} = \frac{1}{\xi^2}(\cosh\tau, \sinh\tau, 0, 0) - \frac{1}{\xi^2}(\cosh\tau, \sinh\tau, 0, 0) = 0, \tag{4.49}$$

it is used $u_{\nu}\dot{a}^{\nu} = -1/\xi^2$.

Therefore, the orbit with $\xi = m/qE = const.$ is an exact solution in which the self-force term vanishes. Isolating the electric field results in

$$E = \frac{m}{q\xi}, \tag{4.50}$$

and, thus,

$$\frac{q}{m}F^{\mu}{}_{\nu} = \frac{1}{\xi}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.51}$$

Note that, as mentioned above, taking the modulus of Equation (4.45), we get, $\xi = 1/|a|$, and the orbits with constant ξ describe exact solutions to the LAD equation, Equation (4.35), for particles with constant proper acceleration when the external force is constant, see, for example, [19], [21].

4.2.1.4. Example: constant external force

In the following, the ORT will be applied to the particular case of the relativistic LAD equation, without the presence of gravitational fields and with constant external force.

Let's proceed with the order reduction applied to Equation (4.35) for a constant electric field E in the x direction. With this condition, the electromagnetic field tensor becomes (4.47). Of course, this fulfills its convergence requirements, since the external source has zero frequency, which is always less than the natural frequencies of the system. To the lowest approximation, neglecting the highest order term, we have

$$a_0^\mu = \frac{q}{m} F^\mu{}_\nu u_0^\nu. \tag{4.52}$$

Consider a time-like $u^\mu = (u^0, u^1, 0, 0)$ with $u_\mu u^\mu = -1$. Then, zeroth-order solution (4.52) is $u_0^\mu = (\cosh(qEt/m), \sinh(qEt/m), 0, 0)$, which coincides with Rindler's orbit (4.44). The derivative of (4.52) substituted into the second term in the right-hand side of (4.35) vanishes, showing that perturbatively the exact result is consistently obtained with the order reduction. This is not anything new, since this exact solution was found by Dirac [21], see also [19]. Similar applications have already been made in [22].

4.2.2. Harmonic oscillator

To emphasize the iterative ORT it will be applied to the harmonic oscillator. In this subsection, both situations with and without external force will be shown for the harmonic oscillator, and then the convergence regimes will be analyzed in both cases.

The equation for the harmonic oscillator is

$$\epsilon \ddot{x} + \gamma \dot{x} + \omega^2 x = f_0 e^{i\Omega t}, \tag{4.53}$$

where $x = x(t)$, $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$, the time dependence of $x(t)$ has been omitted to simplify the notation, f_0 is a constant, γ is the damping coefficient, ω is the natural frequency of the free system, Ω is the external frequency and where, as usual, $0 \leq \epsilon \leq 1$ is a dimensionless perturbative parameter which is set to unity in the end to return to the original equation for the harmonic oscillator.

In order to make possible the application of a perturbative technique, it is necessary to consider the higher order term much smaller than those of lower orders, however, in the equation presented in (4.53) for the harmonic oscillator, it is not possible to do this comparison because the terms x, $\dot{x}$ and $\ddot{x}$ have different dimensions. To solve this problem a dimensionless parameter is defined; $\tau \equiv \gamma t$, the damping constant γ has a dimension of t^{-1}. Thus, $x = x(t) = x(t(\tau))$ and the successive derivatives $\dot{x} = \gamma x'$, $\ddot{x} = \gamma^2 x''$, where $x' = \mathrm{d}x/\mathrm{d}\tau$ and $x'' = \mathrm{d}^2x/\mathrm{d}\tau^2$. As τ is a dimensionless parameter, we can consider the higher order term smaller than the others: $|x''| < |x'| < |x|$, because now they all have the same dimension.

Thus, the Equation (4.53) becomes

$$\epsilon x'' + x' + \frac{\omega^2}{\gamma^2} x = \frac{f_0}{\gamma^2} e^{i(\Omega/\gamma)\tau}. \tag{4.54}$$

Here, the derivatives are with respect to the dimensionless time $\tau \equiv \gamma t$.

The application of the order reduction gives the recurrence relation

$$\epsilon x_n'' + x_{n+1}' + \frac{\omega^2}{\gamma^2} x_{n+1} = \frac{f_0}{\gamma^2} e^{i(\Omega/\gamma)\tau}. \tag{4.55}$$

To lowest order,

$$x_0' + \frac{\omega^2}{\gamma^2}x_0 = \frac{f_0}{\gamma^2}e^{i(\Omega/\gamma)\tau}, \tag{4.56}$$

which can be easily solved assuming $x_0 = c_0 e^{i(\Omega/\gamma)\tau}$, when c_0 is

$$c_0 = \frac{f_0}{\omega^2 + i\gamma\Omega}. \tag{4.57}$$

To first order,

$$\epsilon x_0'' + x_1' + \frac{\omega^2}{\gamma^2}x_1 = \frac{f_0}{\gamma^2}e^{i(\Omega/\gamma)\tau}, \tag{4.58}$$

assuming $x_1 = c_1 e^{i(\Omega/\gamma)\tau}$ and using (4.57) results in

$$c_1 = \frac{f_0 + \epsilon\Omega^2 c_0}{\omega^2 + i\gamma\Omega}. \tag{4.59}$$

Successively,

$$c_{n+1} = \frac{f_0 + \epsilon\Omega^2 c_n}{\omega^2 + i\gamma\Omega}, \tag{4.60}$$

and therefore,

$$\begin{aligned} x_n &= \frac{f_0 e^{i(\Omega/\gamma)\tau}}{\omega^2 + i\gamma\Omega}\left[1 + \frac{\epsilon\Omega^2}{\omega^2 + i\gamma\Omega} + \frac{\epsilon^2\Omega^4}{(\omega^2 + i\gamma\Omega)^2} + \frac{\epsilon^3\Omega^6}{(\omega^2 + i\gamma\Omega)^3} + \ldots\right] \\ &\approx \frac{f_0 e^{i(\Omega/\gamma)\tau}}{\omega^2 + i\gamma\Omega}\left[\frac{1}{1 - \frac{\epsilon\Omega^2}{\omega^2 + i\gamma\Omega}}\right], \end{aligned} \tag{4.61}$$

as long as $\Omega/\omega < 1$, the order reduction, in this case, converges to the exact particular solution

$$x_{n\to\infty} = \frac{f_0 e^{i(\Omega/\gamma)\tau}}{\omega^2 + i\gamma\Omega - \epsilon\Omega^2}, \tag{4.62}$$

for the non homogeneous Equation (4.54). This is the case when there are 2 frequencies, an external one Ω and the natural frequency of the free system ω with $\Omega/\omega < 1$ and this is the situation in which it is very well known the convergence of the order reduction as written in Landau-Lifshitz book [23].

As it is well known, this system has 3 regimes. The underdamped, overdamped, and critically damped. When the system is underdamped $|x''|$ is of the same order of $|x|(\omega/\gamma)^2$. Thus, the order reduction technique does not apply to the underdamped regime.

To apply the order reduction in the homogeneous equation version of Equation (4.54),

$$x_{n+1}' + \frac{\omega^2}{\gamma^2}x_{n+1} = -\epsilon x_n'', \tag{4.63}$$

it is also necessary that $|x''| < |x'| < |x|$. To lowest order

$$x_0' + \frac{\omega^2}{\gamma^2} x_0 = 0. \tag{4.64}$$

Substituting $x_0 = c_0 e^{\lambda_0 \tau}$ into (4.64) gives $\lambda_0 = -\omega^2/\gamma^2$ with

$$x_0 = c_0 e^{-\omega^2 \tau/\gamma^2}. \tag{4.65}$$

To get the next order, x_0 from (4.65) is replaced into

$$\epsilon x_0'' + x_1' + \frac{\omega^2}{\gamma^2} x_1 = 0, \tag{4.66}$$

with solution

$$x_1 = \left(c_1 - \epsilon \frac{\omega^4}{\gamma^4} c_0 \tau\right) \exp\left(-\frac{\omega^2}{\gamma^2}\tau\right), \tag{4.67}$$

shows an additional constant c_1.

To the next orders,

$$x_2 = \left[c_2 - \left(\epsilon c_1 \frac{\omega^4}{\gamma^4} + 2\epsilon^2 c_0 \frac{\omega^6}{\gamma^6}\right)\tau + \left(\epsilon^2 c_1 \frac{\omega^8}{\gamma^8}\right)\frac{\tau^2}{2!}\right] \exp\left(-\frac{\omega^2}{\gamma^2}\tau\right), \tag{4.68}$$

$$\begin{aligned} x_3 = &\left[c_3 - \left(\epsilon c_2 \frac{\omega^4}{\gamma^4} + 2\epsilon^2 c_1 \frac{\omega^6}{\gamma^6} + 5\epsilon^3 c_0 \frac{\omega^8}{\gamma^8}\right)\tau + \left(\epsilon^2 c_1 \frac{\omega^8}{\gamma^8} + 4\epsilon^3 c_0 \frac{\omega^{10}}{\gamma^{10}}\right)\frac{\tau^2}{2!}\right. \\ &\left. - \left(\epsilon^3 c_0 \frac{\omega^{12}}{\gamma^{12}}\right)\frac{\tau^3}{3!}\right] \exp\left(-\frac{\omega^2}{\gamma^2}\tau\right), \end{aligned} \tag{4.69}$$

$$\begin{aligned} x_4 = &\left[c_4 - \left(\epsilon c_3 \frac{\omega^4}{\gamma^4} + 2\epsilon^2 c_2 \frac{\omega^6}{\gamma^6} + 5\epsilon^3 c_1 \frac{\omega^8}{\gamma^8} + 14\epsilon^4 c_0 \frac{\omega^{10}}{\gamma^{10}}\right)\tau + \left(\epsilon^2 c_2 \frac{\omega^8}{\gamma^8} + 4\epsilon^3 c_1 \frac{\omega^{10}}{\gamma^{10}}\right.\right. \\ &\left.\left. + 14\epsilon^4 c_0 \frac{\omega^{12}}{\gamma^{12}}\right)\frac{\tau^2}{2!} - \left(\epsilon^3 c_1 \frac{\omega^{12}}{\gamma^{12}} + 6\epsilon^4 c_0 \frac{\omega^{14}}{\gamma^{14}}\right)\frac{\tau^3}{3!} + \left(\epsilon^4 c_0 \frac{\omega^{16}}{\gamma^{16}}\right)\frac{\tau^4}{4!}\right] \exp\left(-\frac{\omega^2}{\gamma^2}\tau\right), \end{aligned} \tag{4.70}$$

$$\begin{aligned} x_5 = &\left[c_5 - \left(\epsilon c_4 \frac{\omega^4}{\gamma^4} + 2\epsilon^2 c_3 \frac{\omega^6}{\gamma^6} + 5\epsilon^3 c_2 \frac{\omega^8}{\gamma^8} + 14\epsilon^4 c_1 \frac{\omega^{10}}{\gamma^{10}} + 42\epsilon^5 c_0 \frac{\omega^{12}}{\gamma^{12}}\right)\tau\right. \\ &+ \left(\epsilon^2 c_3 \frac{\omega^8}{\gamma^8} + 4\epsilon^3 c_2 \frac{\omega^{10}}{\gamma^{10}} + 14\epsilon^4 c_1 \frac{\omega^{12}}{\gamma^{12}} + 48\epsilon^5 c_0 \frac{\omega^{14}}{\gamma^{14}}\right)\frac{\tau^2}{2!} - \left(\epsilon^3 c_2 \frac{\omega^{12}}{\gamma^{12}} + 6\epsilon^4 c_1 \frac{\omega^{14}}{\gamma^{14}}\right. \\ &\left.\left. + 27\epsilon^5 c_0 \frac{\omega^{16}}{\gamma^{16}}\right)\frac{\tau^3}{3!} + \left(\epsilon^4 c_1 \frac{\omega^{16}}{\gamma^{16}} + 8\epsilon^5 c_0 \frac{\omega^{18}}{\gamma^{18}}\right)\frac{\tau^4}{4!} - \left(\epsilon^5 c_1 \frac{\omega^{20}}{\gamma^{20}}\right)\frac{\tau^5}{5!}\right] \exp\left(-\frac{\omega^2}{\gamma^2}\tau\right). \end{aligned} \tag{4.71}$$

The appearance of additional constants is a direct consequence of this method since, for each perturbative order, a differential equation must be solved. Since higher perturbative orders of whichever perturbative technique must contain the lower order approximations, these additional constants are uniquely determined and made equal to c.

Then, written up to order 5 in ϵ the technique results in

$$x = c\exp\left[-\left(\frac{\omega^2}{\gamma^2}+\epsilon\frac{\omega^4}{\gamma^4}+2\epsilon^2\frac{\omega^6}{\gamma^6}+5\epsilon^3\frac{\omega^8}{\gamma^8}+14\epsilon^4\frac{\omega^{10}}{\gamma^{10}}+42\epsilon^5\frac{\omega^{12}}{\gamma^{12}}\right)\right]\tau. \tag{4.72}$$

Considering Equation (4.63) as a map, in the Appendix D, it is discussed that this map is a contraction, and by the Banach fixed point theorem [76], the method converges to the exact solution

$$x = c\exp\left[\left(-1+\sqrt{1-4\epsilon\omega^2/\gamma^2}\right)\frac{\tau}{2\epsilon}\right], \tag{4.73}$$

which is a fixed point for this map.

In this same Appendix D, it is also discussed that the other fixed point, namely

$$x = c\exp\left[\left(-1-\sqrt{1-4\epsilon\omega^2/\gamma^2}\right)\frac{\tau}{2\epsilon}\right], \tag{4.74}$$

it's not defined when $\epsilon \to 0$. While when $\epsilon \to 0$; both (4.73) has a well defined limit $x = ce^{-\omega^2\tau/\gamma^2}$, which coincides with the exact solution $x = ce^{-\omega^2\tau/\gamma^2}$ of (4.63). This second fixed point, (4.74), then must be excluded and when $\epsilon \to 1$ we are left with the unique solution of (4.63)

$$x = c\exp\left[\left(-1+\sqrt{1-4\omega^2/\gamma^2}\right)\frac{\tau}{2}\right], \tag{4.75}$$

led to conclude that this iterative procedure converges to this solution.

4.3. Starobinsky inflation

We will now apply the order reduction to the case of the Starobinsky inflation [11]. As already mentioned, this inflationary model is the one that best fits the scalar-tensor ratio amplitude, according to the CMBR observations [13], [16], [17].

In this thesis, only the Jordan frame is chosen and the model can be thought of as an effective gravity action that arises naturally as quantum corrections in a consistent model of semi-classical gravity, i.e., in a scenario where quantum matter fields are considered in a classical gravitational background, see, for instance, [12], [39], [40], [41], [42], and [43] the technique developed by Schwinger-DeWitt where the divergences present in the effective action can be eliminated by redefining some renormalized constants [44]. As mentioned in the introduction, (1.1), the necessary counterterms for a consistent theory

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_{PL}^2}{2}\left\{R+\beta R^2+\alpha\left[R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2\right]\right\} \tag{4.76}$$

include Starobinsky inflationary model. In this model α and β are renormalized constants and the cosmological constant is set as null $\Lambda = 0$. The value of $\beta \approx 1.305\times10^9\ {M_{PL}}^{-2}$ can be inferred from observations [13], [14], [15]. Instabilities such as tachyon were pointed out by [33] and [34] for the sign of the regularization parameters $\alpha > 0$ and $\beta < 0$ in the action (4.76).

Metric variations in (4.76), shown in the Appendix E, result in the field equations,

$$E_{\mu\nu} \equiv G_{\mu\nu} + \left(\beta - \frac{1}{3}\alpha\right) H^{(1)}_{\mu\nu} + \alpha H^{(2)}_{\mu\nu} = 0, \tag{4.77}$$

where,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{4.78}$$

$$H^{(1)}_{\mu\nu} = -\frac{R^2}{2} g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box R g_{\mu\nu} - 2R_{;\mu\nu}, \tag{4.79}$$

$$H^{(2)}_{\mu\nu} = -\frac{1}{2} R_{\lambda\sigma} R^{\lambda\sigma} g_{\mu\nu} - R_{;\mu\nu} + 2R^{\lambda\sigma} R_{\lambda\nu\sigma\mu} + \Box R_{\mu\nu} + \frac{1}{2}\Box R g_{\mu\nu}. \tag{4.80}$$

The field equations (4.77) have partial differential equations in order-4 at the metric since the Ricci tensor and the Ricci scalar already includes second-order derivatives in $g_{\mu\nu}$.

For the FLRW homogeneous isotropic line element $g_{\mu\nu} = \mathrm{diag}[-1, a^2, a^2, a^2]$ with zero spatial curvature, there is the 00

$$\frac{1}{6} H^2 + \beta \left[2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2\right] = 0, \tag{4.81}$$

and the 11

$$-\frac{1}{2} H^2 - \frac{1}{3}\dot{H} + \beta \left[-2\dddot{H} - 12\ddot{H}H - 9\dot{H}^2 - 18\dot{H}H^2\right] = 0, \tag{4.82}$$

equations of motion, where $H = \dot{a}/a$ is the Hubble parameter. These equations were obtained using the algebraic manipulator Maple 17. Since the metric is isotropic, $H^{(1)}_{\mu\nu} = 3H^{(2)}_{\mu\nu}$, and the terms that multiply α are canceled. This linear dependence occurs in all conformally flat spacetime, including the FLRW metric, [29] and [35].

It is shown in the Appendix F that the covariant divergence of the equation of motion, (4.77), vanishes, which implies that E_{00} and $E_{0i} = 0$ are constrains, while the dynamic equation is found in the spatial part E_{ij}, see, for instance, [48], [49] and [65]. Thus, the 00 equation of motion, Equation (4.81), as the lowest order equation is a constraint that is dynamically preserved and is used as a numerical check.

As mentioned at the beginning of this chapter and extensively discussed in [66], the field equations from the gravitational $f(R)$ theory results in only a single higher derivative equation which carry the dynamics, Equation (4.82). In this case, it violates the nondegeneracy assumption of the Ostrogradsky instability. The lower order derivative equation results in a constraint that limits the lower derivative degrees of freedom. In Appendix C, the argument is verified for the equation of motion of the Starobinsky inflation, with the FLRW metric employed. In this appendix it was used the conformal equivalence of the gravitational $f(R)$ theory in the Jordan frame with the Einstein-Hilbert gravity in the Einstein frame, see Subsection 3.3.1. Therefore, for the right choice of the sign for the regularization parameter in the Starobinsky model, being of the class of the gravitational $f(R)$ theory, the theory becomes free of tachyon and ghosts.

Again, the order reduction is applied. The temporal equation is chosen for the application of the technique, as it is of the lowest order in H. Since the aim of the order reduction technique is to reduce the ODE by one order, the temporal equation will be reduced from the second to the first order in H. The 00 equation of motion, Equation (4.81), results in the recurrence relation

$$-2\epsilon\beta\frac{\ddot{H}_n}{H_n}+\epsilon\beta\frac{\dot{H}_n^2}{H_n^2}+\frac{1}{6}\left[1-36\beta\dot{H}_{n+1}\right]=0, \tag{4.83}$$

where, as usual, $0\leq\epsilon\leq 1$ is a dimensionless perturbative parameter and at the end is made $\epsilon=1$ to return to the original equation. Here, $H_n\neq 0$ and the equation is dimensionless in the proper time t. The conditions used are $|\beta\ddot{H}|\ll|H|$ and $|\beta\dot{H}^2|\ll|H^2|$. The first and second slow-roll conditions for inflation are given by $|\ddot{H}|\ll|\dot{H}H|$ and $|\dot{H}|\ll|H^2|$, from Equations (3.23) and (3.24), and there is some overlap in convergence region of the order reduction and slow-roll conditions. We mention that it is possible to rewrite the order reduction technique, (4.81), in such a manner that its convergence conditions are identical with the slow-roll conditions. In this case, for the slow-roll conditions, the recurrence relation for the temporal equation is

$$-2\epsilon\beta\frac{\ddot{H}_n}{H_n\dot{H}_n}+\epsilon\beta\frac{\dot{H}_n}{H_n^2}+\frac{1}{6}\left[\frac{1}{\dot{H}_{n+1}}-36\beta\right]=0. \tag{4.84}$$

To lowest order, we have the solution of Ruzmaikina and Ruzmaikin [61], which describes the slow-roll regime

$$1+36\beta\dot{H}_1=0\Rightarrow H_1=\frac{1}{36\beta}(t_{e_1}-t). \tag{4.85}$$

To second order,

$$H_2=\frac{1}{36\beta}(t_{e_2}-t)+\frac{1}{6}\frac{\epsilon}{(t_{e_2}-t)}. \tag{4.86}$$

this solution is also found by [77] for slow-roll conditions.

To third order

$$H_3=\frac{1}{36\beta}(t_{e_3}-t)+\frac{1}{6}\frac{\epsilon}{(t_{e_3}-t)}-\frac{8}{3}\frac{\beta\epsilon^2}{(t_{e_3}-t)^3}. \tag{4.87}$$

The method can be repeated for the next orders.

Until the five order,

$$H_4=\frac{1}{36\beta}(t_{e_4}-t)+\frac{1}{6}\frac{\epsilon}{(t_{e_4}-t)}-\frac{8}{3}\frac{\beta\epsilon^2}{(t_{e_4}-t)^3}+\frac{584}{5}\frac{\beta^2\epsilon^3}{(t_{e_4}-t)^5}, \tag{4.88}$$

$$H_5=\frac{1}{36\beta}(t_{e_5}-t)+\frac{1}{6}\frac{\epsilon}{(t_{e_5}-t)}-\frac{8}{3}\frac{\beta\epsilon^2}{(t_{e_5}-t)^3}+\frac{584}{5}\frac{\beta^2\epsilon^3}{(t_{e_5}-t)^5}-\frac{282048}{35}\frac{\beta^3\epsilon^4}{(t_{e_5}-t)^7}. \tag{4.89}$$

In both cases t_{e_i} is a constant of integration for $i = 1, 2, 3, ...$ and means the end of inflation. A different value is obtained for each t_{e_i} to satisfy the initial condition chosen for H.

As can be shown by the perturbative approximations, Equations (4.85)-(4.89), both conditions for the order reduction and slow-roll are satisfied for

$$1 \ll \frac{1}{\sqrt{\beta}}(t_{e_i} - t). \tag{4.90}$$

In this case, the inflation occurs for $t < t_{e_i}$. While for $t > t_{e_i}$ the conditions are inapplicable, and it presents the graceful exit from inflation.

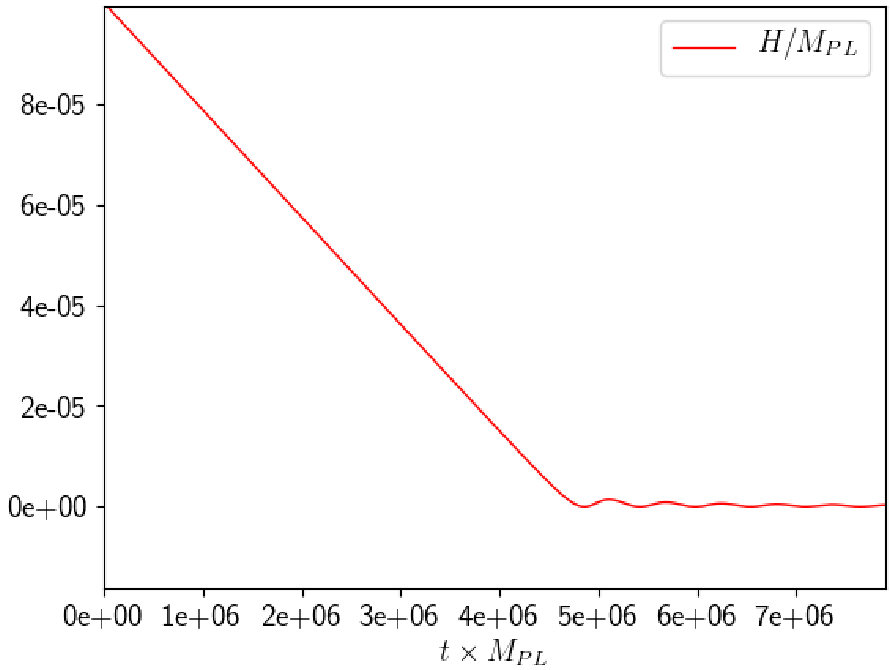

Figure 4.1.: It is well known that $H(t)$ decreases linearly (slow-roll or Ruzmaikina's regime) approaches zero and enters into the phase of the damped oscillations [78]. It is shown in red the exact numerical solution of Eq. (4.81) for $\beta = 1.305 \times 10^9 M_{PL}^{-2}$. The initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12856534 \times 10^{-11} M_{PL}^2$.

It is well known that $H(t)$ decreases linearly (slow-roll or Ruzmaikina's regime) approaches zero and enters into the phase of the damped oscillations [78]. See Figure 4.1, Figure 4.2, Figure 4.3, and Figure 4.4 in red the exact numeric solution of equation (4.81) for $\beta = 1.305 \times 10^9 M_{PL}^{-2}$, $H(t)$. It can be seen that the perturbative solutions do not agree with the field equation (4.81) in the oscillating regime of the weak coupling limit.

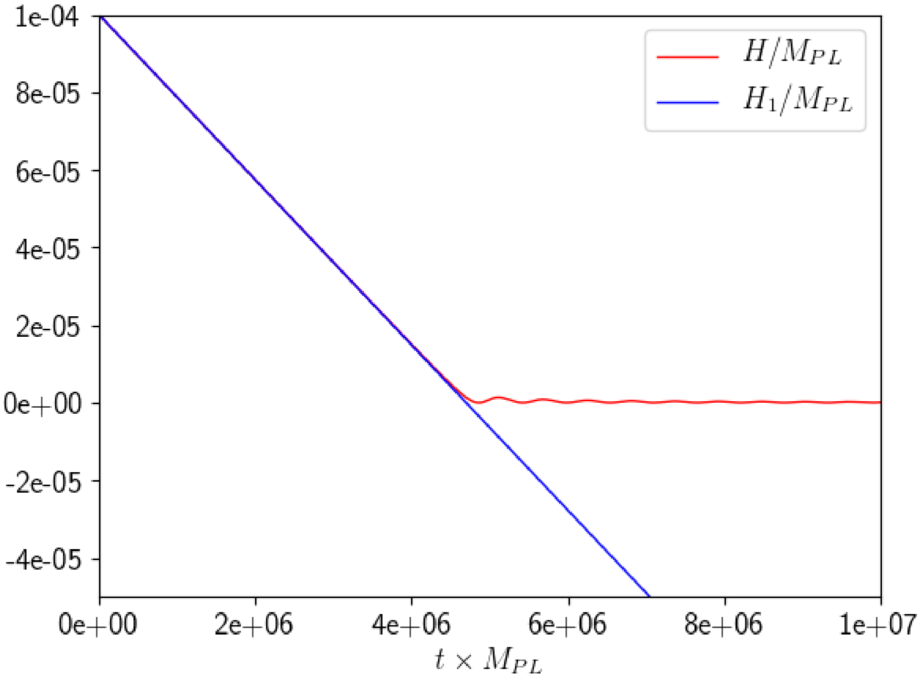

Figure 4.2.: It is shown in red the exact numerical solution of Eq. (4.81) for $\beta = 1.305 \times 10^9 M_{PL}^{-2}$. Plotted in blue is the perturbative solution (4.85), $H_1(t)$. Only one of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12856534 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_1} = 4.69800000 \times 10^6 M_{PL}^{-1}$ in (4.85) are fixed by the choices made. It can be seen that the perturbative solution does not agree with the field equation in the oscillating regime of the weak coupling limit. On the other hand, both solutions show very good agreement in the slow-roll regime.

This is expected, as this region does not fulfill the requirements for the order reduction. On the other hand, both solutions show very good agreement in the slow-roll regime.

As shown in the perturbative approximation (4.89), in this case, the order reduction results in a Laurent series with non-zero principal part with infinite terms [79]. As it's well known this series will not converge in the limit $t \to t_{e_i}$ [80], shown by the asymptotes in Figure 4.1, Figure 4.2 Figure 4.3, and in Figure 4.4. The location of the asymptote in the weak-field limit of small oscillations is a consequence of the value of the constant $t = t_{e_i}$ done exclusively to best fit the initial condition chosen for H. For higher-orders, the asymptotes appear alternated in pairs due to successive powers of β, which must be positive to avoid the tachyon, as mentioned in the introduction.

Besides that, it is possible to see that higher orders of the order reduction method show some convergence to the exact numeric solution as shown in Figure 4.5, Figure 4.6,

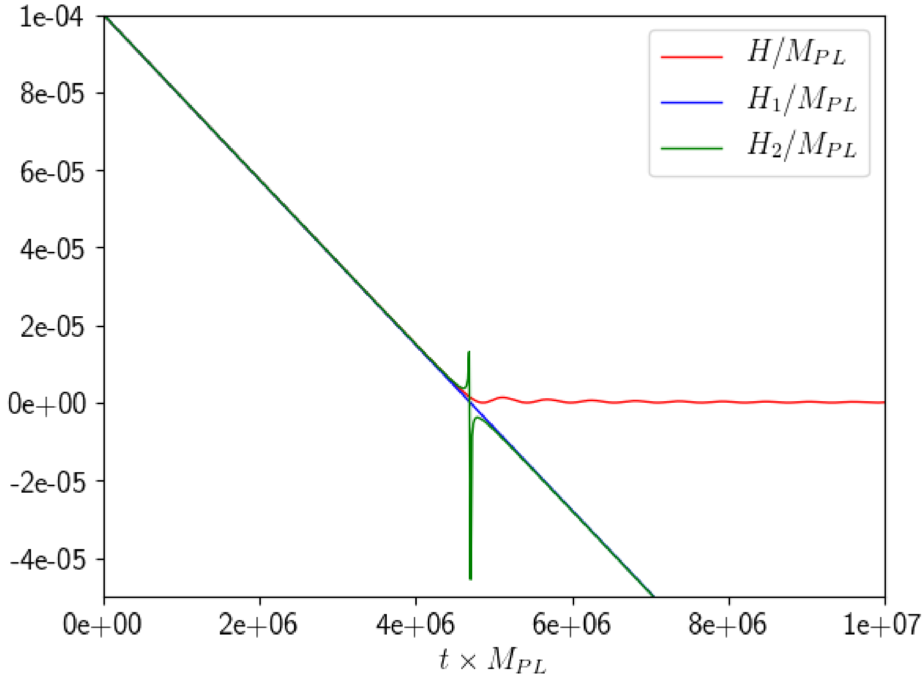

Figure 4.3.: It is shown in red the exact numeric solution of Eq. (4.81) for $\beta = 1.305 \times 10^9 M_{PL}^{-2}$, $H(t)$. The perturbative approximation is plotted in blue for the eq. (4.85), $H_1(t)$. Only one of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12856534 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_1} = 4.69800000 \times 10^6 M_{PL}^{-1}$ are fixed by the choices made. Plotted in green is the perturbative solution for the Eq. (4.86), $H_2(t)$. Only one of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.27810215 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_2} = 4.69633274 \times 10^6 M_{PL}^{-1}$ are fixed by the choices made.

Figure 4.7 and Figure 4.8. It must also be mentioned that the convergence of the order reduction is slow and becomes even slower for higher order approximations.

The situation changes in the presence of sources or spatial curvature since then the field equations will depend explicitly on the scale factor. For instance, in the presence of perfect fluid source $p = w\rho$ with EoS parameter w the covariant conservation of this source $\nabla^\nu T_{\mu\nu} = 0$ implies the Equation (2.11) and a dependence on the scale factor, a, as $\rho = \rho_0(a/a_0)^{-3(w+1)}$. In this case, the method will necessarily present second time derivatives to lowest order instead of first time derivatives in H. If the source is also considered perturbatively, to lowest order, the first derivative equation (4.85) is replaced

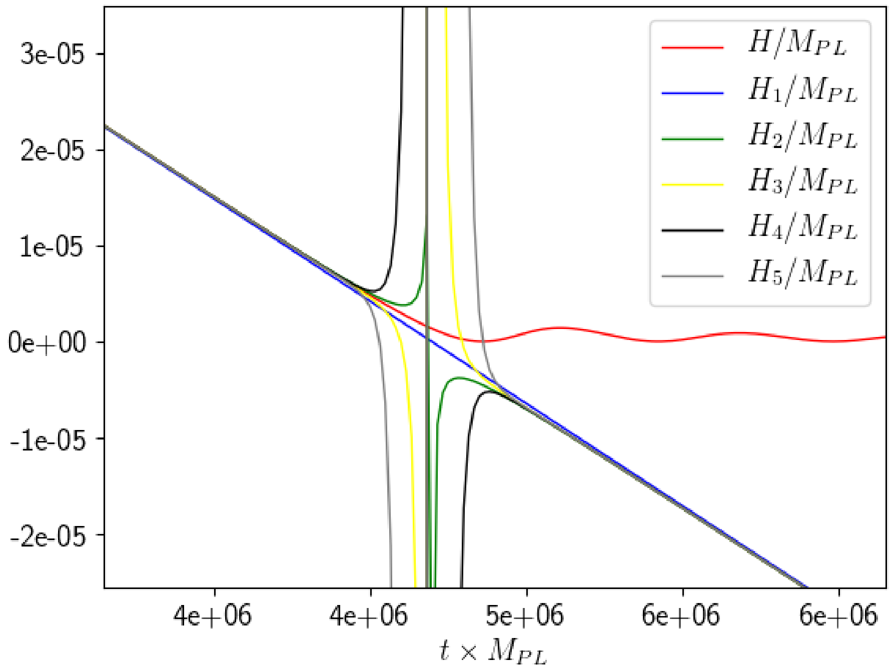

Figure 4.4.: It is shown in red the exact numeric solution of Eq. (4.81) for $\beta = 1.305 \times 10^9 M_{PL}^{-2}$, $H(t)$. The perturbative approximations are plotted in blue for the Eq. (4.85), $H_1(t)$. Only one of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12856534 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_1} = 4.69800000 \times 10^6 M_{PL}^{-1}$ are fixed by the choices made. Plotted in green is the perturbative solution for the Eq. (4.86), $H_2(t)$. Only one of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12781021 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_2} = 4.69633274 \times 10^6 M_{PL}^{-1}$ are fixed by the choices made. In yellow is plotted the perturbative solution for the Eq. (4.87), $H_3(t)$. Only one of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12781182 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_3} = 4.69633432 \times 10^6 M_{PL}^{-1}$ are fixed by choices made. In black is shown the perturbative solution for the Eq. (4.88) $H_4(t)$. One of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_4} = 4.69633431 \times 10^6 M_{PL}^{-1}$ are fixed by the choices made. The last plot in gray shows the perturbative solution for the Eq. (4.89), $H_5(t)$. One of the initial condition is chosen as $H = 1.0 \times 10^{-3} M_{PL}$, while the other $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$ and the constant $t_{e_5} = 4.69633431 \times 10^6 M_{PL}^{-1}$ are fixed by the choices made.

by

$$1 - 36\beta \left[\frac{\ddot{a}_1}{a_1} + \left(\frac{\dot{a}_1}{a_1} \right)^2 \right] = 0, \tag{4.91}$$

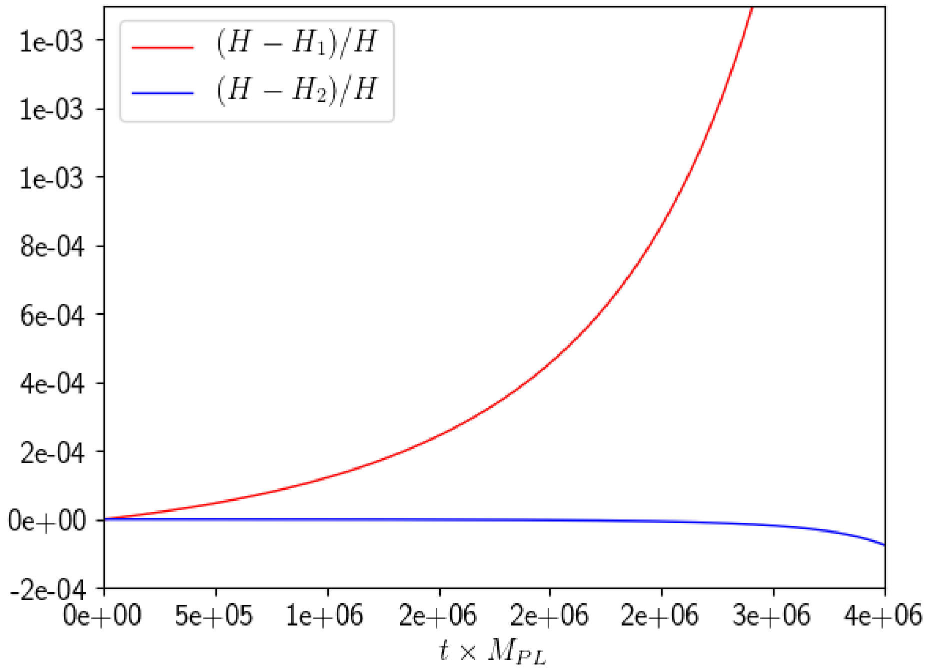

Figure 4.5.: The graph in red shows the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.85), $H_1(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12856534 \times 10^{-11} M_{PL}^2$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_1} = 4.69800000 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.86), $H_2(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781021 \times 10^{-11} M_{PL}^2$, while the constant is $t_{e_2} = 4.69633274 \times 10^6 M_{PL}^{-1}$.

remind that $H = \dot{a}/a$. The explicit dependence of the field equation on the scale factor, through the source $\rho = \rho_0 (a/a_0)^{-3(w+1)}$ will come in higher perturbative approximations, by assumption.

A different choice of variables is presented compared to the work of Simon and Parker [29]. The reason for this is that this order reduction, which we are presenting here, is very sensitive to the choice of the lowest perturbative approximation. The lowest order system in the order reduction must be chosen in accordance with which regime of the solution is going to be reproduced by the method. If higher than first time derivatives of the scale factor are neglected in the lowest perturbative approximation of the order reduction, the Ruzmaikina's regime $H = \dot{a}/a \propto -t$ is not reproduced. For example, in order to get the desired regime, the lowest order must be given by (4.91).

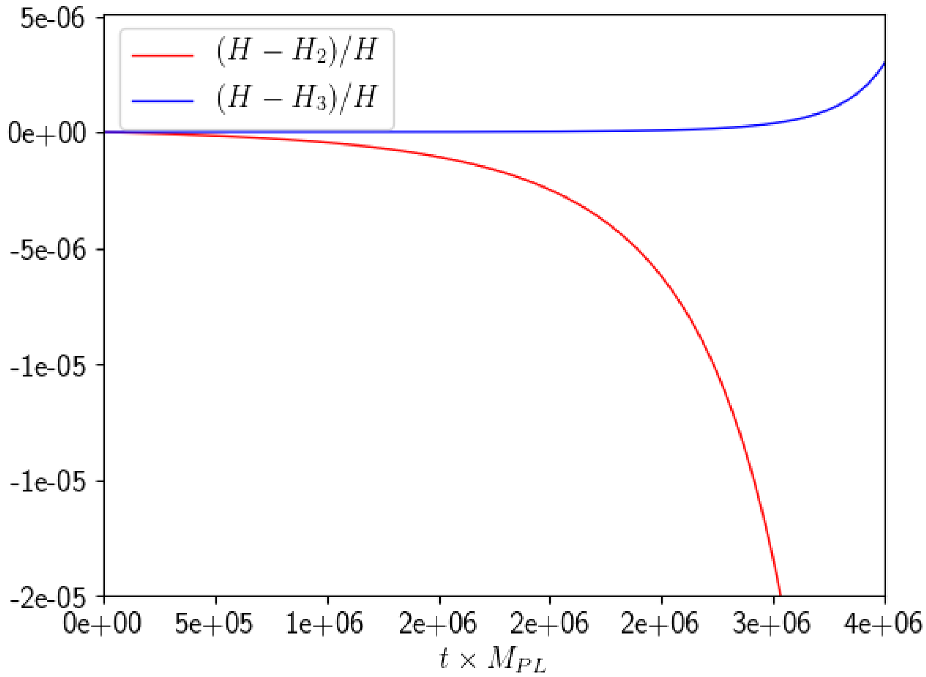

Figure 4.6.: The graph in red shows the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.86), $H_2(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781021 \times 10^{-11} M_{PL}^2$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_2} = 4.69633274 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.87), $H_3(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781182 \times 10^{-11} M_{PL}^2$, while the constant is $t_{e_3} = 4.69633432 \times 10^6 M_{PL}^{-1}$.

To end this chapter, the order reduction will be applied to the trace of the field equation (4.77) for the homogeneous isotropic line element with zero spatial curvature.

The trace of the field equation (4.77) results in

$$g^{\mu\nu}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \left(\beta - \frac{1}{3}\alpha\right)\left(-\frac{R^2}{2}g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box R g_{\mu\nu} - 2R_{;\mu\nu}\right)\right.$$
$$\left. + \alpha\left(-\frac{1}{2}R_{\lambda\sigma}R^{\lambda\sigma}g_{\mu\nu} - R_{;\mu\nu} + 2R^{\lambda\sigma}R_{\lambda\nu\sigma\mu} + \Box R_{\mu\nu} + \frac{1}{2}\Box R g_{\mu\nu}\right)\right] = 0. \tag{4.92}$$

Here, the terms that multiply α are null, the trace (4.92) results in,

$$R - 6\beta\Box R = 0, \tag{4.93}$$

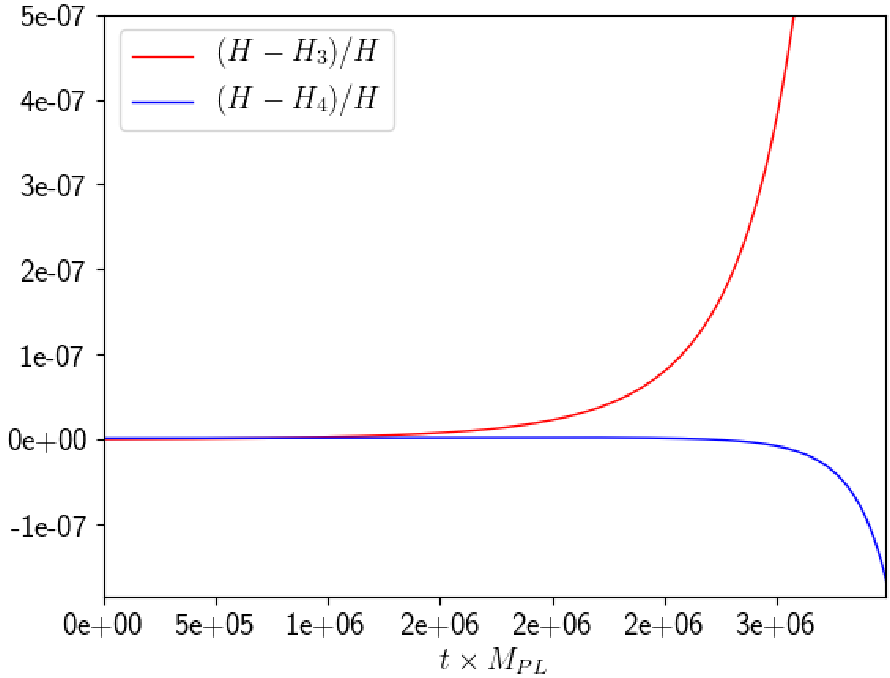

Figure 4.7.: The graph in red shows the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.87), $H_3(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781182 \times 10^{-11} M_{PL}^2$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_3} = 4.69633432 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.88), $H_4(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$, while the constant is $t_{e_4} = 4.69633431 \times 10^6 M_{PL}^{-1}$.

as before, for the FLRW homogeneous isotropic line element $g_{\mu\nu} = \mathrm{diag}[-1, a^2, a^2, a^2]$ with zero spatial curvature, the equation of motion becomes

$$\frac{1}{6}\left[\dot{H} + 2H^2\right] + \beta\left[\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2\right] = 0. \tag{4.94}$$

In following, we apply the ORT to the equation of motion (4.94) and we get the recurrence relation,

$$\frac{\epsilon}{6}\frac{\dot{H}_n}{H_n^2} + \epsilon\beta\left[\frac{\dddot{H}_n}{H_n^2} + 7\frac{\ddot{H}_n}{H_n} + 4\frac{\dot{H}_n^2}{H_n^2}\right] + 12\left[\frac{1}{36} + \beta\dot{H}_{n+1}\right] = 0. \tag{4.95}$$

The conditions used are $\left|\dot{H}\right| \ll \left|H^2\right|$, $|\beta\dddot{H}| \ll \left|H^2\right|$, $\left|\beta\ddot{H}\right| \ll |H|$ and $\left|\beta\dot{H}^2\right| \ll \left|H^2\right|$.

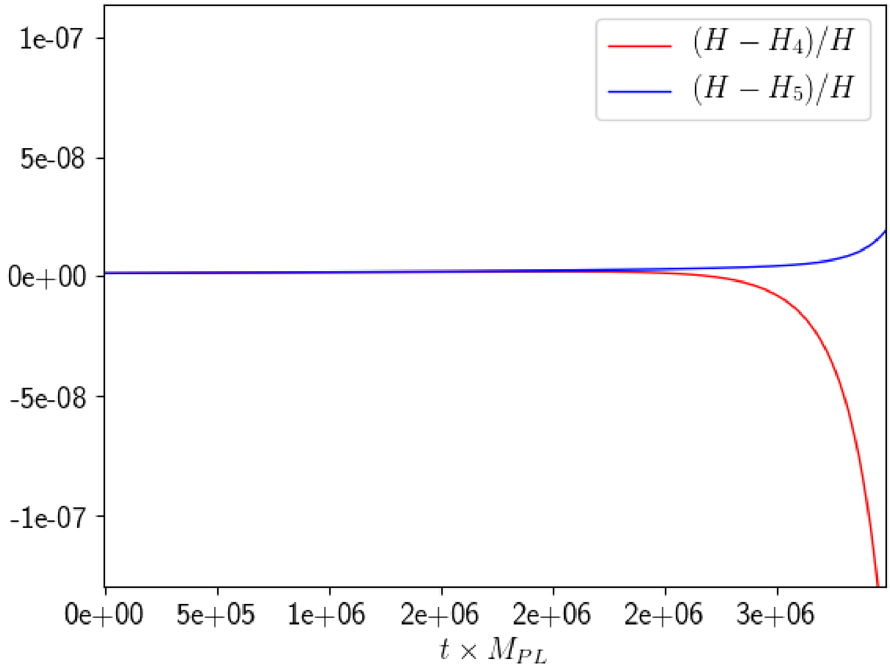

Figure 4.8.: The graph in red shows the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.88), $H_4(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_4} = 4.69633431 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of the difference between the exact numeric solution (4.81), $H(t)$ and the analytical approximation (4.89), $H_5(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$ and $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$, while the constant is $t_{e_5} = 4.69633431 \times 10^6 M_{PL}^{-1}$. It is possible to see a weak convergence for higher orders to the exact numerical solution in the regime of slow-roll.

To lowest order,

$$\frac{1}{36} + \beta \dot{H}_1 = 0 \Rightarrow H_1 = \frac{1}{36\beta}(t_{e_1} - t). \tag{4.96}$$

Until the five order,

$$H_2 = \frac{1}{36\beta}(t_{e_2} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_2} - t)}, \tag{4.97}$$

$$H_3 = \frac{1}{36\beta}(t_{e_3} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_3} - t)} - \left(\frac{8}{3}\frac{\beta}{(t_{e_3} - t)^3} + \frac{108}{5}\frac{\beta^2}{(t_{e_3} - t)^5}\right)\epsilon^2, \tag{4.98}$$

$$H_4 = \frac{1}{36\beta}(t_{e_4} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_4} - t)} - \left(\frac{8}{3}\frac{\beta}{(t_{e_4} - t)^3} + \frac{108}{5}\frac{\beta^2}{(t_{e_4} - t)^5}\right)\epsilon^2 + \left(\frac{692}{5}\frac{\beta^2}{(t_{e_4} - t)^5}\right.$$
$$\left. + \frac{158976}{35}\frac{\beta^3}{(t_{e_4} - t)^7} + 54432\frac{\beta^4}{(t_{e_4} - t)^9}\right)\epsilon^3, \tag{4.99}$$

$$H_5 = \frac{1}{36\beta}(t_{e_5} - t) + \frac{1}{6}\frac{\epsilon}{(t_{e_5} - t)} - \left(\frac{8}{3}\frac{\beta}{(t_{e_5} - t)^3} + \frac{108}{5}\frac{\beta^2}{(t_{e_5} - t)^5}\right)\epsilon^2 + \left(\frac{692}{5}\frac{\beta^2}{(t_{e_5} - t)^5}\right.$$
$$\left. + \frac{158976}{35}\frac{\beta^3}{(t_{e_5} - t)^7} + 54432\frac{\beta^4}{(t_{e_5} - t)^9}\right)\epsilon^3 - \left(\frac{441024}{35}\frac{\beta^3\epsilon^4}{(t_{e_5} - t)^7} + \frac{33642288}{35}\frac{\beta^4}{(t_{e_5} - t)^9}\right.$$
$$\left. + \frac{1769382144}{55}\frac{\beta^5}{(t_{e_5} - t)^{11}} + \frac{5819869440}{13}\frac{\beta^6}{(t_{e_5} - t)^{13}}\right)\epsilon^4. \tag{4.100}$$

The region of convergence is similar with the order reduction and slow-roll conditions applied to the equation (4.81) when is made $\epsilon = 1$ at the end. Again, it can be seen that the successive analytical approximations obtained from the recurrence relation (4.95) show some slow convergence to the exact numerical solution (4.94), as shown in Figure 4.9, Figure 4.10, Figure 4.11 and Figure 4.12.

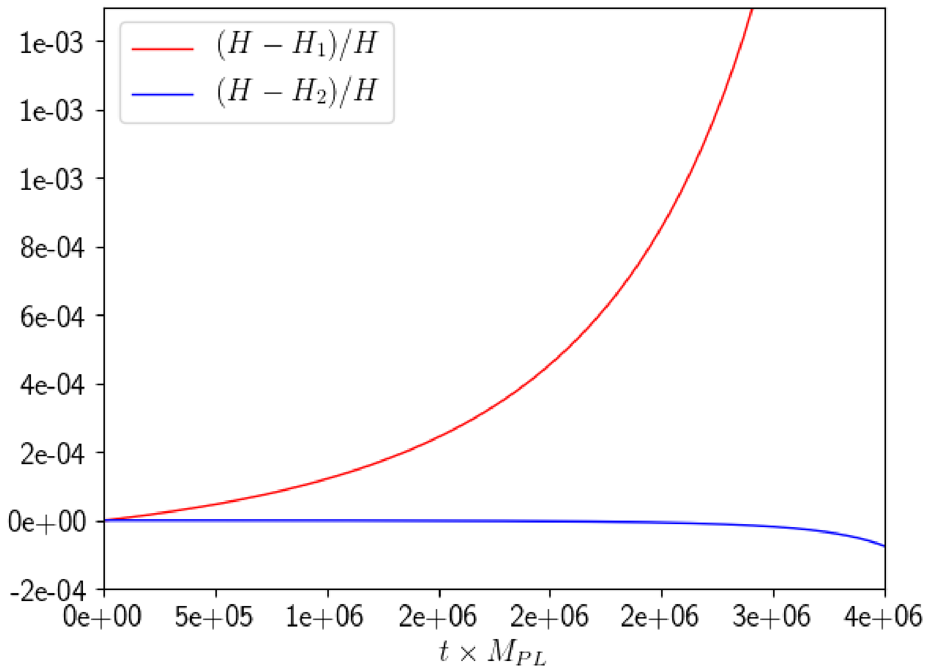

Figure 4.9.: The graph in red shows the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.96), $H_1(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12856534 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 0.0 M_{PL}^3$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_1} = 4.69800000 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.97), $H_2(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781021 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.21469450 \times 10^{-21} M_{PL}^3$, while the constant is $t_{e_2} = 4.69633274 \times 10^6 M_{PL}^{-1}$.

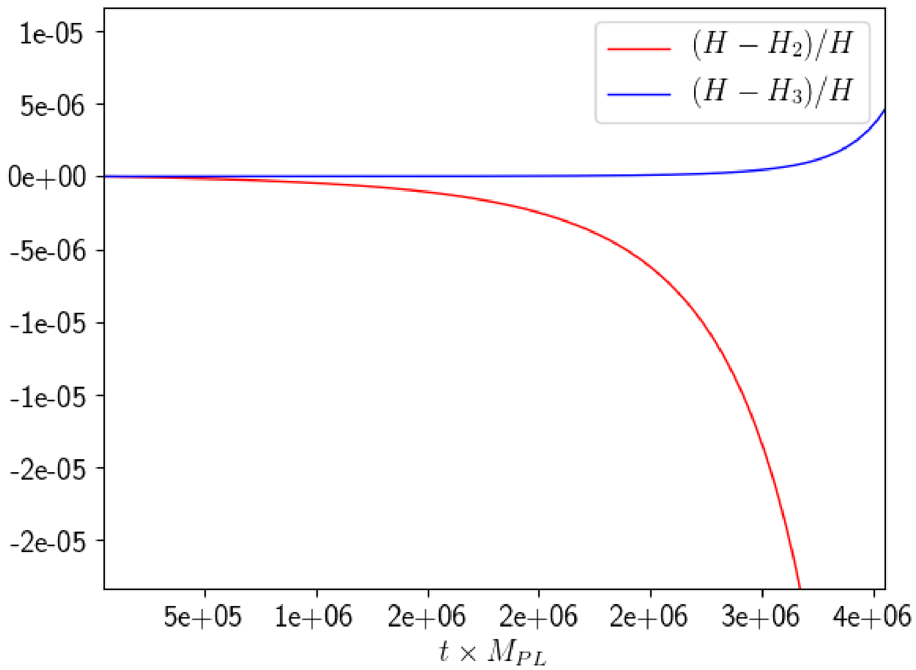

Figure 4.10.: The graph in red shows the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.97), $H_2(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781021 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.21469450 \times 10^{-21} M_{PL}^3$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_2} = 4.69633274 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.98), $H_3(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781182 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.19981479 \times 10^{-21} M_{PL}^3$, while the constant is $t_{e_3} = 4.69633432 \times 10^6 M_{PL}^{-1}$.

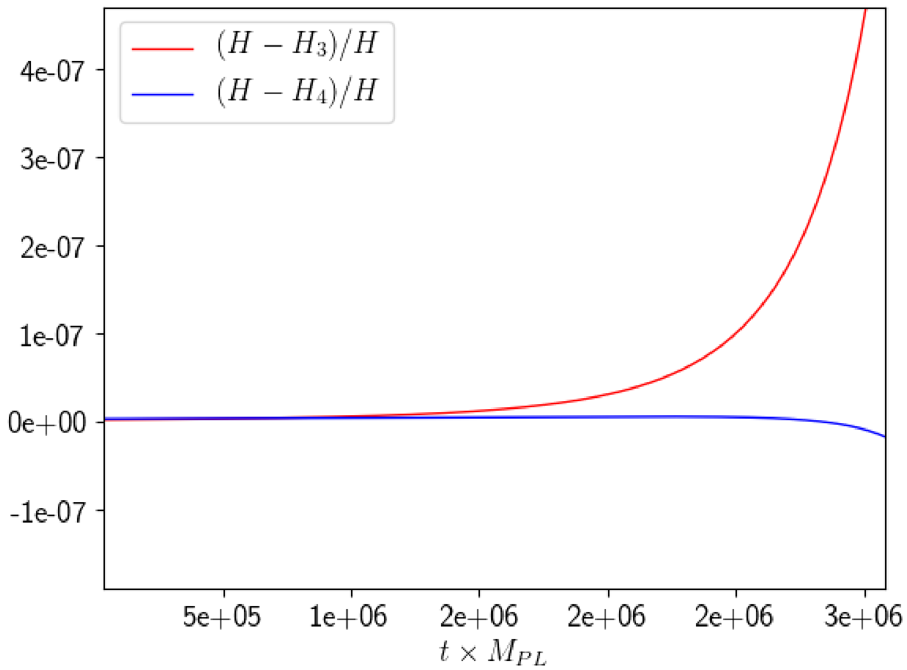

Figure 4.11.: The graph in red shows the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.98), $H_3(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781182 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.19981479 \times 10^{-21} M_{PL}^3$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_3} = 4.69633432 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.99), $H_4(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.19995564 \times 10^{-21} M_{PL}^3$, while the constant is $t_{e_4} = 4.69633431 \times 10^6 M_{PL}^{-1}$.

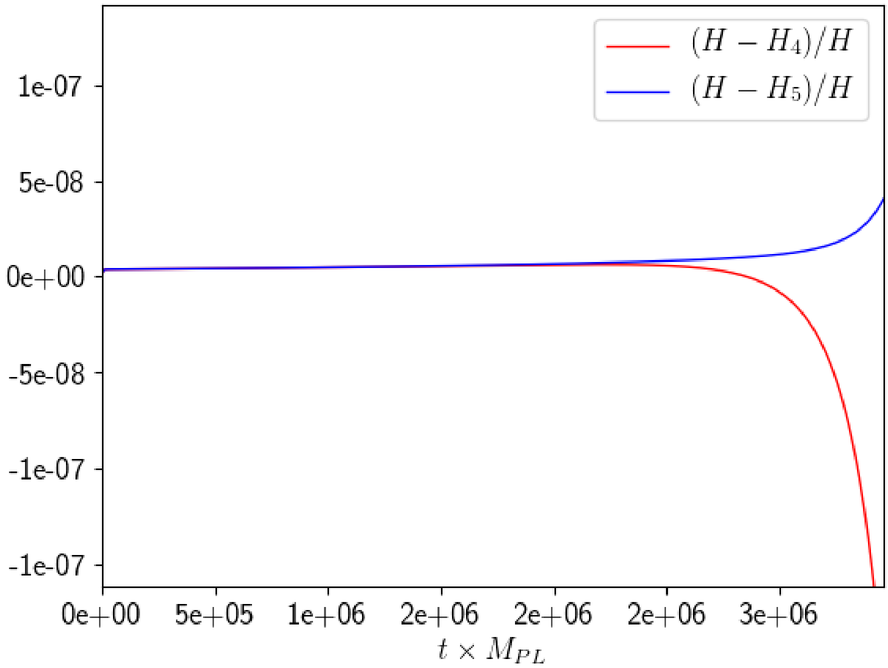

Figure 4.12.: The graph in red shows the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.99), $H_4(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.19995564 \times 10^{-21} M_{PL}^3$. The constants are $\beta = 1.305 \times 10^9 M_{PL}^{-2}$ and $t_{e_4} = 4.69633431 \times 10^6 M_{PL}^{-1}$. Plotted in blue is the ratio of the difference between the exact numeric solution (4.94), $H(t)$ and the analytical approximation (4.100), $H_5(t)$ by the exact numeric solution, $H(t)$. The initial conditions used are $H = 1.0 \times 10^{-3} M_{PL}$, $\dot{H} = -2.12781181 \times 10^{-11} M_{PL}^2$ and $\ddot{H} = 3.19995422 \times 10^{-21} M_{PL}^3$, while the constant is $t_{e_5} = 4.69633431 \times 10^6 M_{PL}^{-1}$.

CHAPTER 5.

Conclusions

In this thesis, it is shown a simple extension of the order reduction as an iterative method of solution of higher order differential equations. The analytical approximations following this technique also are compared with the direct numerical solution of the equation of motion. The main results are based on Ref. [1].

In Section 4.2, some simple examples were presented for which the technique converges to the exact solution. Both situations, with or without a source, are analyzed. Surprisingly, the order reduction presents a very good agreement in strong coupling regimes, non-oscillating which slowly approaches equilibrium. While in the oscillating regime of a weak coupling limit, the ORT is inapplicable. The cases with external sources fall into the class of problems mentioned in the introduction. It is possible to control the external frequency to be much smaller than the natural frequency of the system and order reduction converges to the expected solution.

In Subsection 4.2.1, the order reduction is applied to the relativistic LAD equation of motion in the absence of external gravitational fields. It is considered a constant source and the method gives the well known Rindler motion for the point charge [19]. This is not anything new, since this exact solution can be seen in [19] and some similar applications have already been made in [22]. It must be mentioned that in this situation there is strong coupling and an external source with zero frequency, which is always less than the natural frequencies of the system.

In Subsection 4.2.2, the method is applied to the harmonic oscillator. As expected, the non homogeneous equation converges when submitted to the control of the two frequencies, such that the external frequency must be smaller than the natural frequency of the system. For the homogeneous equation version for the harmonic oscillator, the Banach fixed point theorem is used to show that the order reduction converges uniquely for the overdamped solution of the harmonic oscillator. The oscillating underdamped regime is excluded by the method.

To our knowledge, previous applications of the order reduction to effective gravity mentioned in the introduction [26], [27], [28], [29] seemed to be done only with the presence of sources.

In Section 4.3, the order reduction is applied to Starobinsky inflation [11]. Here, it is considered the model in absence of sources. The order reduction is applied to the field equation (4.81) resulting in the recurrence relation (4.83). The convergence region has some overlap with the first and second slow-roll conditions for inflation. This recurrence relation (4.83) is used to obtain successive analytical approximations that are compared to the direct numerical solution of the equation (4.82). Equation (4.81) as a constraint is dynamically conserved and is used to numerically check the code.

It can be seen in Figure 4.2, Figure 4.3 and Figure 4.4 that the perturbative solution does not agree with the field equation (4.81) in the oscillating weak coupling limit. The asymptote present in the perturbative solutions occurs in the oscillating regime, where the technique of order reduction does not work, as shown by the condition of convergence (4.90). For the following perturbative approximations, the asymptotes appear alternated due to successive powers of β, which must be positive to avoid the tachyon, as mentioned in the introduction. On the other hand, both perturbative solutions showed agreement in the slow-roll regime.

Moreover, it is verified the convergence of the order reduction, as shown in Figure 4.5, Figure 4.6, Figure 4.7 and Figure 4.8 in the slow-roll regime. It is possible to see that successive approximations of the order reduction method show some convergence to the exact numeric solution. It must also be mentioned that this convergence is slow and becomes even slower for higher order approximations.

In the presence of sources or spatial curvature, as mentioned at the end of Section 4.3, the correct choice of the variable should be of scale factor instead of the Hubble parameter. The field equations written with respect to the scale factor will have an additional time derivative as compared to the same field equations written with respect to H as it's done here.

The order reduction presented here is very sensitive to the choice of the lowest perturbative approximation. The lowest order system in the order reduction must be chosen in accordance with which regime of the solution is going to be reproduced by the method. To obtain the desired Ruzmaikina regime, the lowest order for the field equations written in terms of the scale factor must be given by (4.91). If higher than first time derivatives of the scale factor are neglected in the lowest perturbative approximation of the order reduction, the Ruzmaikina's regime $H = \dot{a}/a \propto -t$ is not reproduced.

Furthermore, in Section 4.3, order reduction is also applied to the trace of the field equation (4.77) for the homogeneous isotropic line element with zero spatial curvature used in the previous case of the Starobinsky inflationary model. The equation of motion becomes (4.94) and the order reduction results in the recurrence equation (4.95). The region of convergence is similar with the order reduction and slow-roll conditions applied to the equation (4.81) when is made $\epsilon = 1$ at the end. Again, it is possible to see that higher orders of the order reduction method show some slow convergence to the exact numerical solution (4.94), as shown in Figure 4.9, Figure 4.10, Figure 4.11 and Figure 4.12.

As is well known [81], the order reduced equations present fewer solutions. This was one intention of the ORT to make it easier to select the ones that are physically relevant [26], [29]. This present work agrees with this reasoning. For all solutions analyzed, the perturbative order reduction in its convergence region approaches the physical solutions. However, it must be emphasized that one disadvantage of the method is that there could be some physical solutions that the order reduction will not detect.

Variation on the action for the gravitational f(R) theory

This appendix will show the minimum variation principle of the action of the gravitational $f(R)$ theory

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R), \tag{A.1}$$

in relation to the metric $g_{\mu\nu}$.

First note that

$$\delta[\sqrt{-g}f] = f\delta\sqrt{-g} + \sqrt{-g}f'\delta R, \tag{A.2}$$

where for simplicity of notation $f(R) \equiv f$ and $f' \equiv \partial f/\partial R$.

To calculate the variation of this action the following equations in the free fall frame become useful:

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \tag{A.3}$$

$$\delta g^{\mu\nu} = -g^{\mu\sigma}g^{\nu\lambda}\delta g_{\sigma\lambda}, \tag{A.4}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{A.5}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{A.6}$$

$$R^{\mu}{}_{\nu\lambda\sigma} = \Gamma^{\mu}_{\nu\sigma,\lambda} - \Gamma^{\mu}_{\nu\lambda,\sigma}, \tag{A.7}$$

$$R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma} = \Gamma^{\mu}_{\nu\sigma,\mu} - \Gamma^{\mu}_{\nu\mu,\sigma}, \tag{A.8}$$

$$\delta\Gamma^{\sigma}_{\mu\nu} = \frac{g^{\sigma\lambda}}{2}\left[\delta g_{\lambda\mu,\nu} + \delta g_{\lambda\nu,\mu} - \delta g_{\mu\nu,\lambda}\right]. \tag{A.9}$$

See, for instance, [48] and [49].

Using Equations (A.8) and (A.9) in the second term of Equation (A.6) follows

$$\int d^4x\sqrt{-g}f'g^{\mu\nu}\delta R_{\mu\nu} = \int d^4x\sqrt{-g}\frac{f'}{2}g^{\mu\nu}[g^{\sigma\lambda}\left(\delta g_{\lambda\mu,\nu\sigma} + \delta g_{\lambda\nu,\mu\sigma} - \delta g_{\mu\nu,\lambda\sigma}\right)$$
$$- g^{\sigma\lambda}\left(\delta g_{\lambda\mu,\sigma\nu} + \delta g_{\lambda\sigma,\mu\nu} - \delta g_{\mu\sigma,\lambda\nu}\right)] = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}[g^{\sigma\lambda}(f'_{,\nu\sigma}\delta g_{\lambda\mu} + f'_{,\mu\sigma}\delta g_{\lambda\nu}$$
$$- f'_{,\lambda\sigma}\delta g_{\mu\nu} - f'_{,\sigma\nu}\delta g_{\lambda\mu} + f'_{,\mu\nu}\delta g_{\lambda\sigma} - f'_{,\lambda\nu}\delta g_{\mu\sigma})] = \frac{1}{2}\int d^4x\sqrt{-g}(f'^{\mu\nu}_{,}\delta g_{\mu\nu} - \Box f' g^{\mu\nu}\delta g_{\mu\nu}$$
$$- \Box f' g^{\mu\nu}\delta g_{\mu\nu} + f'^{\mu\nu}_{,}\delta g_{\mu\nu}) = \int d^4x\sqrt{-g}(f'^{\mu\nu}_{,}\delta g_{\mu\nu} - \Box f' g^{\mu\nu}\delta g_{\mu\nu}), \tag{A.10}$$

where Leibniz's rule is used, and the surface integral terms vanish.

Substituting Equations (A.3), (A.4), (A.6) and (A.10) in Equation (A.2) the variation of the action for the $f(R)$ theory results in

$$\delta S = \frac{1}{2\kappa}\int d^4x\delta[\sqrt{-g}f] = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}f\delta g^{\mu\nu} + f'R_{\mu\nu}\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu f'\delta g^{\mu\nu}\right.$$
$$\left.+\Box f' g_{\mu\nu}\delta g^{\mu\nu}\right) = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}f + f'R_{\mu\nu} - \nabla_\mu\nabla_\nu f' + \Box f' g_{\mu\nu}\right)\delta g^{\mu\nu} \tag{A.11}$$

where Equation (A.4) is being used to move the metric index up and down within the variation.

By the minimum variation principle $\delta S = 0$ and the term inside the parentheses cancels out, resulting in the equation of motion,

$$f'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f - \nabla_\mu\nabla_\nu f' + \Box f' g_{\mu\nu} = 0. \tag{A.12}$$

Conformal transformation

The conformal transformation from the Jordan frame to the Einstein frame is as follows

$$\tilde{g}_{\mu\nu} = e^{\Omega} g_{\mu\nu}, \tag{B.1}$$
$$\tilde{g}^{\mu\nu} = e^{-\Omega} g^{\mu\nu}. \tag{B.2}$$

Where e^{Ω} is the conformal factor and the tilde represents the quantity in the Einstein frame.

From the Jordan's frame in free fall

$$\Gamma^{\mu}_{\nu\lambda} = 0, \qquad \Gamma^{\mu}_{\nu\lambda,\sigma} \neq 0, \tag{B.3}$$

the definition (A.9) for the connection, and the conformal transformations (B.1)-(B.2), we have

$$\begin{aligned}\tilde{\Gamma}^{\mu}_{\nu\lambda} &= \frac{1}{2}\tilde{g}^{\mu\gamma}[\tilde{g}_{\nu\gamma,\lambda} + \tilde{g}_{\lambda\gamma,\nu} - \tilde{g}_{\nu\lambda,\gamma}] = \frac{1}{2}e^{-\Omega}g^{\mu\gamma}\left[\left(e^{\Omega}g_{\nu\gamma}\right)_{,\lambda} + \left(e^{\Omega}g_{\lambda\gamma}\right)_{,\nu} - \left(e^{\Omega}g_{\nu\lambda}\right)_{,\gamma}\right] \\ &= \Gamma^{\mu}_{\nu\lambda} + \frac{1}{2}g^{\mu\gamma}\left[g_{\nu\gamma}\Omega_{,\lambda} + g_{\lambda\gamma}\Omega_{,\nu} - g_{\nu\lambda}\Omega_{,\gamma}\right].\end{aligned} \tag{B.4}$$

According to (B.3) the first term on the right-hand side vanishes and therefore

$$\tilde{\Gamma}^{\mu}_{\nu\lambda} = \frac{1}{2}g^{\mu\gamma}\left[g_{\nu\gamma}\Omega_{,\lambda} + g_{\lambda\gamma}\Omega_{,\nu} - g_{\nu\lambda}\Omega_{,\gamma}\right]. \tag{B.5}$$

Substituting the connection in the Einstein frame written in terms of the Jordan frame, (B.5), in the Riemann tensor

$$\tilde{R}^{\mu}_{\nu\lambda\sigma} = \tilde{\Gamma}^{\mu}_{\nu\sigma,\lambda} - \tilde{\Gamma}^{\mu}_{\nu\lambda,\sigma} + \tilde{\Gamma}^{\mu}_{\gamma\lambda}\tilde{\Gamma}^{\gamma}_{\nu\sigma} - \tilde{\Gamma}^{\mu}_{\gamma\sigma}\tilde{\Gamma}^{\gamma}_{\nu\lambda}, \tag{B.6}$$

results in,

$$\begin{aligned}\tilde{R}^{\mu}_{\nu\lambda\sigma} = &\left[\frac{1}{2}g^{\mu\gamma}\left(g_{\nu\gamma}\Omega_{,\sigma} + g_{\sigma\gamma}\Omega_{,\nu} - g_{\nu\sigma}\Omega_{,\gamma}\right)\right]_{,\lambda} + \frac{1}{4}g^{\mu\epsilon}\left(g_{\gamma\epsilon}\Omega_{,\lambda} + g_{\lambda\epsilon}\Omega_{,\gamma} - g_{\gamma\lambda}\Omega_{,\epsilon}\right) \\ &g^{\gamma\theta}\left(g_{\nu\theta}\Omega_{,\sigma} + g_{\sigma\theta}\Omega_{,\nu} - g_{\nu\sigma}\Omega_{,\theta}\right) + \Gamma^{\mu}_{\nu\sigma,\lambda} - (\lambda \leftrightarrow \sigma).\end{aligned} \tag{B.7}$$

Using $g^{\mu\nu}{}_{,\lambda} = 0$ and $g^{\mu\lambda} g_{\nu\lambda} = \delta^{\mu}_{\nu}$ in Equation (B.7) the Riemann tensor written in the Einstein frame becomes

$$\tilde{R}^{\mu}_{\nu\lambda\sigma} = R^{\mu}{}_{\nu\lambda\sigma} + \frac{1}{2}\left(\delta^{\mu}_{\nu}\Omega_{,\sigma\lambda} + \delta^{\mu}_{\sigma}\Omega_{,\nu\lambda} - g^{\mu\gamma} g_{\nu\sigma}\Omega_{,\gamma\lambda}\right) + \frac{1}{4}\left(\delta^{\mu}_{\gamma}\Omega_{,\lambda} + \delta^{\mu}_{\lambda}\Omega_{,\gamma} - g^{\mu\epsilon} g_{\gamma\lambda}\Omega_{,\epsilon}\right)$$
$$\left(\delta^{\gamma}_{\nu}\Omega_{,\sigma} + \delta^{\gamma}_{\sigma}\Omega_{,\nu} - g^{\gamma\theta} g_{\nu\sigma}\Omega_{,\theta}\right) - (\lambda \leftrightarrow \sigma). \tag{B.8}$$

The Ricci tensor is obtained through (B.8) as

$$\tilde{R}_{\nu\sigma} = \tilde{R}^{\mu}_{\nu\mu\sigma} = R_{\nu\sigma} + \frac{1}{2}\left[\Omega_{,\sigma\nu} + \Omega_{,\nu\sigma} - g_{\nu\sigma}\Box\Omega\right] - \frac{1}{2}\left[\Omega_{,\nu\sigma} + 4\Omega_{,\nu\sigma} - \Omega_{,\nu\sigma}\right]$$
$$+ \frac{1}{4}\left[\left(\Omega_{,\gamma} + 4\Omega_{,\gamma} - \Omega_{,\gamma}\right)\left(\delta^{\gamma}_{\nu}\Omega_{,\sigma} + \delta^{\gamma}_{\sigma}\Omega_{,\nu} - g_{\nu\sigma}\Omega_{,}{}^{\gamma}\right) - \left(\delta^{\mu}_{\gamma}\Omega_{,\sigma} + \delta^{\mu}_{\sigma}\Omega_{,\gamma} - g_{\gamma\sigma}\Omega_{,}{}^{\mu}\right)\right.$$
$$\left.\left(\delta^{\gamma}_{\nu}\Omega_{,\mu} + \delta^{\gamma}_{\mu}\Omega_{,\nu} - g_{\nu\mu}\Omega_{,}{}^{\gamma}\right)\right]. \tag{B.9}$$

Manipulations on Equation (B.9) result in

$$\tilde{R}_{\nu\sigma} = R_{\nu\sigma} - \Omega_{,\nu\sigma} - \frac{1}{2} g_{\nu\sigma}\Box\Omega + \frac{1}{2}\Omega_{,\nu}\Omega_{,\sigma} - \frac{1}{2} g_{\nu\sigma}\Omega_{,\gamma}\Omega_{,}{}^{\gamma}. \tag{B.10}$$

To end, the Ricci scalar is obtained through the Ricci tensor (B.10) and the conformal transformation according to (B.2) as

$$\tilde{R} = \tilde{R}^{\mu}_{\mu} = \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu} = e^{-\Omega} g^{\mu\nu}\left(R_{\mu\nu} - \Omega_{,\mu\nu} - \frac{1}{2} g_{\mu\nu}\Box\Omega + \frac{1}{2}\Omega_{,\mu}\Omega_{,\nu} - \frac{1}{2} g_{\mu\nu}\Omega_{,\gamma}\Omega_{,}{}^{\gamma}\right). \tag{B.11}$$

Manipulations on (B.11) result in the following expression for the Ricci scalar in the Einstein frame in terms of the Jordan frame,

$$\tilde{R} = e^{-\Omega}\left(R - 3\Box\Omega - \frac{3}{2}\Omega_{,\gamma}\Omega_{,}{}^{\gamma}\right). \tag{B.12}$$

Conformal equivalence of the f(R) theory and the Einstein-Hilbert gravity

As mentioned in the Subsection 3.3.1, the gravitational $f(R)$ theory is conformally equivalent to the Einstein-Hilbert gravity. In the original Jordan frame, the scalar field ϕ is non-minimally coupled with the Ricci scalar. It is possible to rewrite this theory in the Einstein frame with the scalar field coupled with the matter by the conformal transformation of the metric tensor (3.37) and with a re-scaling of the scalar field, see Equation (3.46).

Thus, the action for the gravitational $f(R)$ theory, see Equation (3.33), nonlinear in the Ricci scalar, becomes the action (3.47) given in the Einstein frame. This results in the Einstein-Hilbert equation $G_{\mu\nu} = T_{\mu\nu}(\tilde{\phi})$ and in the equation of classic motion for the scalar field (3.13), according to the choice and definition of (3.39) and (3.46), see [60]. As a particular case with the FLWR metric employed, these equations are:

$$3\tilde{H}^2 = \kappa \left(\frac{1}{2}\tilde{\phi}'^2 + V(\tilde{\phi}) \right), \tag{C.1}$$

$$2\tilde{H}' + 3\tilde{H}^2 = -\kappa \left(\frac{1}{2}\tilde{\phi}'^2 - V(\tilde{\phi}) \right), \tag{C.2}$$

$$\tilde{\phi}'' + 3\tilde{H}\tilde{\phi}' + \frac{\mathrm{d}V(\tilde{\phi})}{\mathrm{d}\tilde{\phi}} = 0, \tag{C.3}$$

see Equations (2.14) and (2.17) for the first and second Friedmann equations, Equations (3.6) and (3.7) for the temporal and spatial terms of the energy-momentum tensor. Here the prime means a time derivative in the Einstein frame and this time dependence has been omitted to simplify the notation.

In this appendix, it will be shown how these equations lead to only a single higher derivative equation that carries the dynamics and a single lower order derivative equation that results in a constraint in the Jordan frame.

Through the conformal transformation as mentioned earlier, $\tilde{g}_{\mu\nu} = e^{\Omega} g_{\mu\nu}$, with the choice $e^{\Omega} = f'$, where as before $f(R) \equiv f$ and $f' \equiv \partial f / \partial R$, follows the re-definitions

$$d\tilde{t} = \sqrt{f'}dt, \qquad \tilde{a}(\tilde{t}) = \sqrt{f'}a(t). \tag{C.4}$$

The Hubble parameter and its first time derivative in the Einstein frame becomes,

$$\tilde{H} = \frac{\tilde{a}'}{\tilde{a}} = \frac{H}{\sqrt{f'}}\left(1 + \frac{\dot{R}f''}{2Hf'}\right), \tag{C.5}$$

$$\tilde{H}' = \frac{1}{\sqrt{f'}}\left(\frac{\dot{H}}{\sqrt{f'}} + \frac{\ddot{R}f'' + \dot{R}^2 f''' - H\dot{R}f''}{2f'^{3/2}} - \frac{3}{4}\frac{(\dot{R}f'')^2}{f'^{5/2}}\right). \tag{C.6}$$

Here $f'' \equiv \partial f'/\partial R$, $f''' \equiv \partial f''/\partial R$ and the dot means a time derivative in the Jordan frame.

As seen before, the re-scaling scalar field Equation (3.46), and its first and second time derivative in the Einstein frame are

$$\tilde{\phi} = \sqrt{\frac{3}{2\kappa}}\ln f', \tag{C.7}$$

$$\tilde{\phi}' = \sqrt{\frac{3}{2\kappa}}\frac{f''\dot{R}}{f'^{3/2}}, \tag{C.8}$$

$$\tilde{\phi}'' = \frac{1}{\sqrt{f'}}\sqrt{\frac{3}{2\kappa}}\left(\frac{f'''\dot{R}^2 + f''\dot{R}}{f'^{3/2}} - \frac{3}{2}\frac{(\dot{R}f'')^2}{f'^{5/2}}\right). \tag{C.9}$$

The potential and its scalar field derivative in the Einstein frame are

$$V(\tilde{\phi}) = \frac{Rf' - f}{2\kappa f'^2}, \tag{C.10}$$

$$\frac{\mathrm{d}V(\tilde{\phi})}{\mathrm{d}\tilde{\phi}} = \sqrt{\frac{2\kappa}{3}}\frac{1}{4\beta\kappa}\left(\frac{f' - 1}{f'^2}\right). \tag{C.11}$$

Using Equations (C.5), (C.8) and (C.10) in Equation (C.1), it results in

$$3H^2 = \frac{\kappa}{f''}\left(\frac{Rf'' - f}{2\kappa} - \frac{3Hf''\dot{R}}{\kappa}\right). \tag{C.12}$$

Replacing Equations (C.5), (C.6) and (C.8), (C.10) in Equation (C.2), we obtain

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f''}\left(\frac{\dot{R}^2 f''' + (2H\dot{R} + \ddot{R})f''}{\kappa} - \frac{Rf'' - f}{2\kappa}\right). \tag{C.13}$$

Finally, substituting Equations (C.5), (C.8), (C.9) and (C.11) in Equation (C.3) follows

$$3\dot{R}f''H + f'''\dot{R}^2 + f''\ddot{R} + \frac{1}{6\beta}(f' - 1) = 0. \tag{C.14}$$

Equations (C.12)-(C.14) for Starobinsky inflation $f(R) = R + \beta R^2$ in the Jordan frame follow only a single higher order derivative dynamic equation with a constraint given by the temporal lower order derivative equation:

$$\frac{1}{6}H^2 + \beta\left[2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2\right] = 0, \tag{C.15}$$

$$-\frac{1}{2}H^2 - \frac{1}{3}\dot{H} + \beta\left[-2\dddot{H} - 12\ddot{H}H - 9\dot{H}^2 - 18\dot{H}H^2\right] = 0. \tag{C.16}$$

These same equations can be found by the principle of minimum variation in Appendix E, with the FLRW employed in the Section 4.3, see Equations (4.81)-(4.82).

Convergence of the order reduction to harmonic oscillator

Consider the following first order differential equation for x_{n+1} to be understood as an iteration map

$$x'_{n+1} + \frac{\omega^2}{\gamma^2} x_{n+1} = -\epsilon x''_n, \tag{D.1}$$

where the parameter $0 \le \epsilon \le 1$ and at the end is made $\epsilon = 1$. We will check that the above iteration map is a contraction. Variables are changed to $x_n = e^{-\omega^2\tau/\gamma^2} y_n$ assuming that y_n and its derivatives are limited functions in the time interval in question

$$I = [\tau_0, \tau]. \tag{D.2}$$

Integrating by parts twice

$$\begin{aligned}
&y'_{n+1} = -\epsilon e^{\omega^2\tau/\gamma^2} x''_n, \\
&y_{n+1} - y^0_{n+1} = -\epsilon \int_{\tau_0}^{\tau} e^{\omega^2 s/\gamma^2} x''_n(s) ds, \\
&y_{n+1} - y^0_{n+1} = -\epsilon \left[e^{\omega^2 s/\gamma^2} x'_n(s) \right]_{\tau_0}^{\tau} + \epsilon \frac{\omega^2}{\gamma^2} \left[e^{\omega^2 s/\gamma^2} x_n(s) \right]_{\tau_0}^{\tau} \\
&- \epsilon \frac{\omega^4}{\gamma^4} \int_{\tau_0}^{\tau} e^{\omega^2 s/\gamma^2} x_n(s) ds, \\
&y_{n+1} - y^0_{n+1} = -\epsilon \left[y'_n(s) - 2\frac{\omega^2}{\gamma^2} y_n(s) \right]_{\tau_0}^{\tau} - \epsilon \frac{\omega^4}{\gamma^4} \int_{\tau_0}^{\tau} y_n(s) ds,
\end{aligned} \tag{D.3}$$

where y^0_{n+1} is the initial condition. The metric is induced by the uniform norm

$$||y(\tau)|| = \sup_{\tau \in I} |y(\tau)|. \tag{D.4}$$

First, we show that for a given function y, its first iteration y_1 is within some upper limit,

$$||y_1 - y^0_1|| < b, \quad b > 0. \tag{D.5}$$

Begin with

$$y_1 - y_1^0 = -\epsilon \left[y'(s) - 2\frac{\omega^2}{\gamma^2} y(s) \right]_{\tau_0}^{\tau} - \epsilon \frac{\omega^4}{\gamma^4} \int_{\tau_0}^{\tau} y(s) ds, \tag{D.6}$$

from (D.3). Then, considering that both

$$\begin{aligned} |y(\tau) - y(\tau_0)| &\leq \sup \left| \frac{dy}{d\tau} \right| \Delta\tau = \|y'\| \Delta\tau \\ |y'(\tau) - y'(\tau_0)| &\leq \sup \left| \frac{d^2y}{d\tau^2} \right| \Delta\tau = \|y''\| \Delta\tau \end{aligned} \tag{D.7}$$

and that

$$\left| \int_{\tau_0}^{\tau} y(s) ds \right| \leq \int_{\tau_0}^{\tau} |y(s)| ds \leq \|y\| \Delta\tau \tag{D.8}$$

with $\Delta\tau = \tau - \tau_0$ it is possible to rewrite $\|y_1 - y^0\|$ as

$$\|y_1 - y_1^0\| \leq \epsilon \left\{ \|y''\| + 2\frac{\omega^2}{\gamma^2} \|y'\| + \frac{\omega^4}{\gamma^4} \|y\| \right\} \Delta\tau. \tag{D.9}$$

Since y and its derivatives have a definite norm, it is always possible to choose $\Delta\tau$ such that $\|y_1 - y_1^0\| < b$.

Now, given two functions y_a and y_b, we shall prove that

$$\|y_a^1 - y_b^1\| \leq q \|y_a - y_b\|, \tag{D.10}$$

for some $0 \leq q < 1$. We will suppose that both y' and y'' have Lipschitz constants $L_1 \geq 0$ and $L_2 \geq 0$ in the time interval I

$$\|y_a' - y_b'\| \leq L_1 \|y_a - y_b\| \qquad\qquad \|y_a'' - y_b''\| \leq L_2 \|y_a - y_b\| \tag{D.11}$$

which is a rather strong condition, anyway reasonable, since, by assumption, all these functions are limited in the considered time interval. Following (D.6) for two distinct functions y_a and y_b with same initial condition $y_a^0 = y_b^0$ and performing their difference results in

$$\begin{aligned} y_a^1 - y_b^1 = &-\epsilon \left[y_a'(s) - y_b'(s) - 2\frac{\omega^2}{\gamma^2} \left(y_a(s) - y_b(s) \right) \right]_{\tau_0}^{\tau} \\ &- \epsilon \frac{\omega^4}{\gamma^4} \int_{\tau_0}^{\tau} (y_a(s) - y_b(s)) ds. \end{aligned} \tag{D.12}$$

Keeping in mind (D.7) and (D.8) then

$$\|y_a^1 - y_b^1\| \leq \epsilon \left\{ L_2 + 2\frac{\omega^2}{\gamma^2} L_1 + \frac{\omega^4}{\gamma^4} \right\} \|y_a - y_b\| \Delta\tau. \tag{D.13}$$

It is always possible to choose a sufficiently small time interval $\Delta\tau$ such that the above relation is

$$\|y_a^1 - y_b^1\| \leq q\|y_a - y_b\| \tag{D.14}$$

with $0 \leq q < 1$, which shows that the map is a contraction in the metric space of functions with uniform norm. Banach fixed point theorem states that, since it is a contraction map, it has a unique fixed point [76].

It must be mentioned since the analytic solution is known, that there are two fixed points x_∞^+ and x_∞^- for (D.1)

$$x_\infty^\pm = c\exp\left[\left(-1 \pm \sqrt{1 - \frac{4\epsilon\omega^2}{\gamma^2}}\right)\frac{\tau}{2\epsilon}\right]. \tag{D.15}$$

We know that x_∞^+ is one of the solutions of (D.1). We call it x_n and take its second derivative x_n'' to get x_{n+1}. Substituting x_n'' in (D.1) we get the following ODE for x_{n+1}

$$\epsilon\left[-\frac{1}{2\epsilon} + \frac{1}{2\epsilon}\sqrt{1 - 4\frac{\omega^2}{\gamma^2}}\right]^2 \exp\left\{\left[-\frac{1}{2\epsilon} + \frac{1}{2\epsilon}\sqrt{1 - 4\frac{\omega^2}{\gamma^2}}\right]t\right\} + x_{n+1}' + \frac{\omega^2}{\gamma^2}x_{n+1} = 0, \tag{D.16}$$

solving this ODE, we get that $x_{n+1} = x_\infty^+$. The same proof follows for the other solution of (D.1), namely x_∞^-. Thus, we show that x_∞^+ and x_∞^- are the two fixed points that (D.1) has. On the other hand, it can be easily seen in (D.1) that when $\epsilon \to 0$ the solution is $x_n = e^{-\omega^2\tau/\gamma^2}$. Now, only one of the fixed points $x_\infty^\pm$ is consistent with this solution, namely,

$$x_\infty^+ = c\lim_{\epsilon=0}\left\{\exp\left[\left(-1 + \sqrt{1 - \frac{4\epsilon\omega^2}{\gamma^2}}\right)\frac{\tau}{2\epsilon}\right]\right\} = e^{-\omega^2\tau/\gamma^2}. \tag{D.17}$$

The other fixed point, x_∞^- does not have a well defined limit when $\epsilon \to 0$ and must be excluded.

The iterative procedure (D.1), when $\epsilon \to 1$, then converges to the unique solution

$$x = c\exp\left[\left(-1 + \sqrt{1 - \frac{4\omega^2}{\gamma^2}}\right)\frac{\tau}{2}\right]. \tag{D.18}$$

Effective action variation

For the effective gravity, the action is

$$S = \int \mathrm{d}^4x \sqrt{-g} \frac{M_{PL}^2}{2} \left\{ R + \beta R^2 + \alpha \left[R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right] \right\}. \tag{E.1}$$

By the minimum variation principle, the effective gravity action variation must be zero. To obtain this variation, the Equations (A.3)-(A.9) together with

$$\partial \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial g_{\mu\nu}, \tag{E.2}$$

$$\partial g^{\mu\nu} = -g^{\mu\sigma} g^{\nu\lambda} \partial g_{\sigma\lambda}, \tag{E.3}$$

become useful.

For the linear curvature term, we have

$$\int \mathrm{d}^4x \delta(\sqrt{-g} R) = \int \mathrm{d}^4x \delta(\sqrt{-g} g^{\mu\nu} R_{\mu\nu}) \tag{E.4}$$

$$= \int \mathrm{d}^4x \left\{ (\delta\sqrt{-g}) R + \sqrt{-g} (\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} (\delta R_{\mu\nu}) \right\}$$

$$= \int \mathrm{d}^4x \left\{ -\frac{1}{2} \sqrt{-g} R g_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} (\delta R_{\mu\nu}) \right\} \tag{E.5}$$

the last term becomes null, because

$$\int \mathrm{d}^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int \mathrm{d}^4x \sqrt{-g} g^{\mu\nu} \delta(\Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu})$$

$$= \int \mathrm{d}^4x \left\{ \partial_\lambda [\sqrt{-g} g^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\nu}] - \partial_\nu [\sqrt{-g} g^{\mu\nu} \delta\Gamma^{\lambda}_{\mu\lambda}] \right\} = 0 \tag{E.6}$$

where we use Equations (E.2)-(E.3), Leibniz's rule, and the fact that the surface integral terms vanish.

Thus,

$$\int \mathrm{d}^4x \delta(\sqrt{-g} R) = \int \mathrm{d}^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu}. \tag{E.7}$$

This term, when considered alone in action, results in the Einstein-Hilbert field equation. See, for instance, [48] and [49].

For the quadratic curvature term, we have

$$\int \mathrm{d}^4x\delta(\sqrt{-g}R^2) = \int \mathrm{d}^4x\delta(\sqrt{-g}R_{\mu\lambda}g^{\mu\lambda}R_{\nu\sigma}g^{\nu\sigma})$$
$$= \int \mathrm{d}^4x \left\{\sqrt{-g}\left(-\frac{R^2}{2}g_{\mu\nu} + 2RR_{\mu\nu}\right)\delta g^{\mu\nu} + \sqrt{-g}2Rg^{\mu\nu}\delta R_{\mu\nu}]\right\}, \tag{E.8}$$

where Equations (A.3)-(A.4) are used. The last term becomes

$$\int \mathrm{d}^4x\sqrt{-g}2Rg^{\mu\nu}\delta R_{\mu\nu} = \int \mathrm{d}^4x\sqrt{-g}2Rg^{\mu\nu}\delta(\Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu})$$
$$= \int \mathrm{d}^4x\sqrt{-g}\left(-2R_{;\lambda}g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} + 2R_{;\nu}g^{\mu\nu}\Gamma^{\lambda}_{\mu\lambda}\right)$$
$$= \int \mathrm{d}^4x\sqrt{-g}\left\{-R_{;}{}^{\gamma}g^{\mu\nu}[(\delta g_{\gamma\mu})_{;\nu} + (\delta g_{\gamma\nu})_{;\mu} - (\delta g_{\mu\nu})_{;\gamma}] + R_{;}{}^{\mu}g^{\lambda\gamma}[(\delta g_{\gamma\mu})_{;\lambda} + (\delta g_{\gamma\lambda})_{;\mu}\right.$$
$$\left.-(\delta g_{\mu\lambda})_{;\gamma}]\right\} = \int \mathrm{d}^4x\sqrt{-g}(-2\Box Rg^{\mu\nu} + 2R_{;}{}^{\mu\nu})\delta g_{\mu\nu}, \tag{E.9}$$

where (A.8)-(A.9) and Leibniz's rule with zero surface integral terms are used. As we are working in the free fall reference frame, we can always change the partial derivatives (comma), for covariant derivatives (semicolon), because in this reference the connection is null. Substituting (E.9) into (E.8) results in

$$\int \mathrm{d}^4x\delta(\sqrt{-g}R^2) = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{R^2}{2}g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box Rg_{\mu\nu} - 2R_{;\mu\nu}\right)\delta g^{\mu\nu}, \tag{E.10}$$

with (A.4) being used to move the metric index up and down within the variation.

In the end, for the product of the Ricci tensors, we have

$$\int \mathrm{d}^4x\delta(\sqrt{-g}R_{\mu\nu}R^{\mu\nu})$$
$$= \int \mathrm{d}^4x\sqrt{-g}\left\{\left[-\frac{1}{2}R^{\sigma\nu}R_{\sigma\nu}g_{\mu\lambda} + 2R_{\mu}{}^{\sigma}R_{\lambda\sigma}\right]\delta g^{\mu\lambda} + 2R^{\mu\nu}\delta R_{\mu\nu}\right\}, \tag{E.11}$$

where Equations (A.3)-(A.4) are used. The last term becomes

$$\int \mathrm{d}^4x\sqrt{-g}R^{\mu\nu}\delta R_{\mu\nu}$$
$$= \int \mathrm{d}^4x\sqrt{-g}\frac{1}{2}\left(-2R_{\nu\lambda;\mu}{}^{\nu} + \Box R_{\mu\lambda} + R_{\lambda\nu;}{}^{\nu}{}_{\mu} + R_{\nu\sigma;}{}^{\nu\sigma}g_{\mu\lambda} - R_{\mu\nu;}{}^{\nu}{}_{\lambda}\right)\delta g^{\mu\lambda}, \tag{E.12}$$

proceeding in the same way as in the Equation (E.9).

By the relation between commutators of the covariant derivative

$$R_{\nu\lambda;\mu}{}^{\nu} = R_{\nu\lambda;}{}^{\nu}{}_{\mu} + R^{\sigma}{}_{\mu}R_{\sigma\lambda} + R_{\lambda}{}^{\sigma\nu}{}_{\mu}R_{\nu\sigma}, \tag{E.13}$$

and using, $R^{\mu\nu}{}_{;\mu} = R^{\mu\nu}{}_{,\mu} + (Rg^{\mu\nu})_{,\mu}$, we have

$$R^{\mu\nu}{}_{;\mu} = \frac{1}{2}R_{,\nu}. \tag{E.14}$$

Thus,

$$\int \mathrm{d}^4x\delta(\sqrt{-g}R_{\mu\nu}R^{\mu\nu}) \tag{E.15}$$

$$= \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}R_{\lambda\sigma}R^{\lambda\sigma}g_{\mu\nu} - R_{;\mu\nu} + 2R^{\lambda\sigma}R_{\lambda\nu\sigma\mu} + \Box R_{\mu\nu} + \frac{1}{2}\Box Rg_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{E.16}$$

Therefore, the equation of motion becomes

$$E_{\mu\nu} \equiv G_{\mu\nu} + \left(\beta - \frac{1}{3}\alpha\right)H^{(1)}_{\mu\nu} + \alpha H^{(2)}_{\mu\nu} = 0, \tag{E.17}$$

with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{E.18}$$

$$H^{(1)}_{\mu\nu} = -\frac{R^2}{2}g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box Rg_{\mu\nu} - 2R_{;\mu\nu}, \tag{E.19}$$

$$H^{(2)}_{\mu\nu} = -\frac{1}{2}R_{\lambda\sigma}R^{\lambda\sigma}g_{\mu\nu} - R_{;\mu\nu} + 2R^{\lambda\sigma}R_{\lambda\nu\sigma\mu} + \Box R_{\mu\nu} + \frac{1}{2}\Box Rg_{\mu\nu}. \tag{E.20}$$

Covariant divergence

In the following, it is verified that the covariant divergence of the equation of motion (E.17) is zero.

For the first term, which is the Einstein-Hilbert gravity (E.18),

$$\bigtriangledown^{\mu} G_{\mu\nu} = \bigtriangledown^{\mu} R_{\mu\nu} - \frac{1}{2} \bigtriangledown^{\mu} (g_{\mu\nu} R) = R_{\mu\nu;}{}^{\mu} - \frac{1}{2}(g_{\mu\nu} R_{;}{}^{\mu}) = \frac{1}{2} R_{;\nu} - \frac{1}{2} R_{;\nu} = 0, \tag{F.1}$$

with Equation (E.14) and the metric condition $\bigtriangledown_{\mu} g^{\mu\nu} = 0$ used.

For the second term (E.19),

$$\begin{aligned} \bigtriangledown^{\mu} H^{(1)}_{\mu\nu} &= \bigtriangledown^{\mu} \left(-\frac{R^2}{2} g_{\mu\nu} + 2RR_{\mu\nu} + 2\Box R g_{\mu\nu} - 2R_{;\mu\nu} \right) \\ &= (-\bigtriangledown^{\mu} R g_{\mu\nu}) R + 2(\bigtriangledown^{\mu} R) R_{\mu\nu} + 2R(\bigtriangledown^{\mu} R_{\mu\nu}) + 2 \bigtriangledown^{\mu} \Box R g_{\mu\nu} - 2 \bigtriangledown^{\mu} R_{;\mu\nu} \\ &= -RR_{;}\nu + 2R_{;}{}^{\mu} R_{\mu\nu} + RR_{;}\nu + 2\Box R_{;}{}^{\nu} - 2R_{;\mu} R^{\mu}{}_{\nu} - 2\Box R_{;}{}^{\nu} \\ &= 0, \end{aligned} \tag{F.2}$$

where it is used $R^{ab}{}_{;a} = \frac{1}{2} R_{,b}$, the metricity condition and the following relation for the commutators of the covariant derivative

$$\bigtriangledown_{\nu} \Box R = \bigtriangledown_{\nu} \bigtriangledown^{\mu} (\bigtriangledown_{\mu} R) = \bigtriangledown^{\mu} \bigtriangledown_{\nu} \bigtriangledown_{\mu} R + R^{\mu}{}_{\lambda\mu\nu} \bigtriangledown^{\lambda} R = \Box R_{;\nu} - R^{\lambda}{}_{\nu} \bigtriangledown_{\lambda} R. \tag{F.3}$$

Finally, for the last term (E.20),

$$\begin{aligned} \bigtriangledown^{\mu} H^{(2)}_{\mu\nu} &= \bigtriangledown^{\mu} \left(-\frac{1}{2} R_{\lambda\sigma} R^{\lambda\sigma} g_{\mu\nu} - R_{;\mu\nu} + 2R^{\lambda\sigma} R_{\lambda\nu\sigma\mu} + \Box R_{\mu\nu} + \frac{1}{2} \Box R g_{\mu\nu} \right) \\ &= -R_{\lambda\sigma;\nu} R^{\lambda\sigma} - \Box R_{;\nu} + 2R^{\lambda\sigma}{}_{;}{}^{\mu} R_{\lambda\nu\sigma\mu} + 2R^{\lambda\sigma} R_{\lambda\nu\sigma\mu;}{}^{\mu} + \bigtriangledown^{\mu} \bigtriangledown^{\sigma} \bigtriangledown_{\sigma} R_{\mu\nu} \\ &+ \frac{1}{2} \bigtriangledown_{\nu} \bigtriangledown^{\sigma} \bigtriangledown_{\sigma} R = R^{\lambda\sigma} \left(R_{\lambda\nu;\sigma} - R_{\lambda\sigma;\nu} + R_{\lambda\nu\sigma\mu;}{}^{\mu} \right), \end{aligned} \tag{F.4}$$

where it is used

$$\bigtriangledown^{\mu} \bigtriangledown^{\sigma} \bigtriangledown_{\sigma} R_{\mu\nu} = R_{\mu\lambda;}{}^{\sigma} R_{\nu}{}^{\lambda\mu}{}_{\sigma} + \frac{\Box R_{;\nu}}{2} + \bigtriangledown_{\sigma} \left[R^{\lambda\sigma} R_{\lambda\nu} + R_{\nu}{}^{\lambda\mu\sigma} R_{\mu\lambda} \right] \tag{F.5}$$

and Equation (F.3). Substituting Bianchi's identity $R_{\lambda\nu\sigma\mu} = R_{\sigma\lambda\mu\nu} + R_{\lambda\mu\sigma\nu}$ in (F.4) and observing that $R^{\lambda\sigma}R_{\sigma\lambda\mu\nu} = 0$ we get

$$\bigtriangledown^{\mu} H^{(2)}_{\mu\nu} = R^{\lambda\sigma}\left(R_{\lambda\nu;\sigma} - R_{\lambda\sigma;\nu} + R_{\sigma\nu\lambda\nu;}{}^{\mu}\right) = 0, \tag{F.6}$$

where the Jacobi identity is used $R_{\lambda\nu\gamma\mu;\sigma} + R_{\lambda\nu\mu\sigma;\gamma} + R_{\lambda\nu\sigma\gamma;\mu} = 0$.

Thus, it is shown the following covariant divergence

$$\bigtriangledown^{\mu} E_{\mu\nu} = 0, \tag{F.7}$$

for $E_{\mu\nu}$ given by Equation (E.17).

Bibliography

[1] W. P. F. De Medeiros and D. Müller. *On the order reduction*. Eur. Phys. J. C **81** 231 (2021). `arXiv:2009.00629`. Cited on pages iv, v, vi, 4, 5, 26, and 54.

[2] S. Profumo. *An Introduction to Particle Dark Matter*. World Scientific 2017. Cited on page 1.

[3] P. C. VI. *Planck 2018 results. VI. Cosmological parameters*. Astron. Astrophys. **641** A6 (2020). `arXiv:1807.06209`. Cited on pages 1 and 11.

[4] S. Dodelson. *Modern cosmology*. Academic Press San Diego, CA (2003). Cited on page 1.

[5] D. Baumann. *Inflation*. In *Physics of the large and the small, TASI 09, proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado, USA, 1-26 June 2009* pages 523–686 2011. `arXiv:0907.5424`. Cited on pages 1, 11, 13, and 17.

[6] E. W. Kolb and M. S. Turner. *The early universe*. Frontiers in Physics. Westview Press Boulder, CO (1990). Cited on pages 1, 14, and 17.

[7] V. Mukhanov. *Physical Foundations of Cosmology*. Cambridge Univ. Press Cambridge 2005. Cited on pages 1, 10, 18, 20, and 21.

[8] B. A. Bassett, S. Tsujikawa, and D. Wands. *Inflation dynamics and reheating*. Rev. Mod. Phys. **78** 537 (2006). Cited on page 1.

[9] M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby. *General Relativity: An Introduction for Physicists*. Cambridge University Press 2006. Cited on page 1.

[10] J. Martin, C. Ringeval, and V. Vennin. *Encyclopædia Inflationaris*. Phys. Dark Univ. **5-6** 75 (2014). `arXiv:1303.3787`. Cited on page 1.

[11] A. A. Starobinsky. *A New Type of Isotropic Cosmological Models Without Singularity*. Phys. Lett. **B91** 99 (1980). Cited on pages 1, 2, 4, 5, 38, and 54.

[12] B. DeWitt. *Dynamical Theory of Groups and Fields*. Documents on modern physics. Gordon and Breach 1965, cap.17. Cited on pages 2, 5, 30, and 38.

[13] Y. Akrami et al. *Planck 2018 results - X. Constraints on inflation.* A&A **641** A10 (2020). Cited on pages 2, 19, and 38.

[14] D. S. Gorbunov and A. G. Panin. *Are R^2- and Higgs-inflations really unlikely?* Physics Letters B **743** 79 (2015). arXiv:1412.3407. Cited on pages 2 and 38.

[15] S. S. Mishra, V. Sahni, and A. V. Toporensky. *Initial conditions for Inflation in an FRW Universe.* Phys. Rev. **D98** 083538 (2018). arXiv:1801.04948. Cited on pages 2 and 38.

[16] S. Hanany et al. *PICO: Probe of Inflation and Cosmic Origins.* Bull. Am. Astron. Soc. **51** 194 (2019). Cited on pages 2, 18, 19, and 38.

[17] F. R. Bouchet et al. *COrE (Cosmic Origins Explorer) A White Paper.* hal-00730257 **v1** (2011). Cited on pages 2, 19, and 38.

[18] M. Abraham. *Theorie der Elektrizität: Elektromagnetische Theorie der Strahlung* volume 2. Teubner 1905. Cited on pages 2 and 30.

[19] F. Rohrlich. *Classical charged particles.* World Scientific Publishing Co Inc 2007. Cited on pages 2, 26, 29, 32, 33, 34, 35, and 54.

[20] M. Abraham. *Prinzipien der Dynamik des Elektrons.* Annalen der Physik **315** 105 (1902). Cited on pages 2 and 30.

[21] P. A. Dirac. *Classical theory of radiating electrons.* Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences pages 148–169 (1938). Cited on pages 3, 25, 30, 32, 33, 34, and 35.

[22] C. J. Eliezer and R. E. Peierls. *On the classical theory of particles.* Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **194** 543 (1948). Cited on pages 3, 4, 31, 35, and 54.

[23] L. D. Landau and E. M. Lifshitz. *The classical theory of fields.* Pergamon 1951. Cited on pages 4, 25, 31, and 36.

[24] E. Poisson, A. Pound, and I. Vega. *The motion of point particles in curved spacetime.* Living Reviews in Relativity **14** 7 (2011). Cited on pages 4 and 30.

[25] S. E. Gralla, A. I. Harte, and R. M. Wald. *Rigorous derivation of electromagnetic self-force.* Physical Review D **80** 024031 (2009). Cited on page 4.

[26] L. Bel and H. S. Zia. *Regular reduction of relativistic theories of gravitation with a quadratic Lagrangian.* Phys. Rev. **D32** 3128 (1985). Cited on pages 4, 54, and 55.

[27] J. Z. Simon. *Stability of flat space, semiclassical gravity, and higher derivatives.* Phys. Rev. D **43** 3308 (1991). Cited on pages 4 and 54.

[28] J. Z. Simon. *No Starobinsky inflation from self-consistent semiclassical gravity.* Phys. Rev. D **45** 1953 (1992). Cited on pages 4, 5, and 54.

[29] L. Parker and J. Z. Simon. *Einstein equation with quantum corrections reduced to second order*. Phys. Rev. **D47** 1339 (1993). `arXiv:gr-qc/9211002`. Cited on pages 4, 5, 25, 39, 45, 54, and 55.

[30] P. R. Anderson. *Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe*. Phys. Rev. D **28** 271 (1983). Cited on pages 4 and 5.

[31] P. R. Anderson. *Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe. II*. Phys. Rev. D **29** 615 (1984). Cited on pages 4 and 5.

[32] P. R. Anderson. *Effects of Quantum Fields on Singularities and Particle Horizons in the Early Universe. 4. Initially Empty Universes*. Phys. Rev. D **33** 1567 (1986). Cited on pages 4 and 5.

[33] B. Whitt. *Stability of Schwarzschild black holes in fourth-order gravity*. Phys. Rev. D **32** 379 (1985). Cited on pages 5 and 38.

[34] D. Müller, M. E. Alves, and J. C. de Araujo. *The Isotropization Process in the Quadratic Gravity*. Int. J. Mod. Phys. D **23** 1450019 (2014). Cited on pages 5 and 38.

[35] V. Ginzburg, D. Kirzhnits, and A. Lyubushin. *The Role of Quantum Fluctuations of the Gravitational Field in General Relativity Theory and Cosmology*. Soviet Journal of Experimental and Theoretical Physics **33** 242 (1971). Cited on pages 5 and 39.

[36] P. Davies, S. Fulling, S. Christensen, and T. Bunch. *Energy Momentum Tensor of a Massless Scalar Quantum Field in a Robertson-Walker Universe*. Annals Phys. **109** 108 (1977). Cited on page 5.

[37] R. M. Wald. *The Back Reaction Effect in Particle Creation in Curved Space-Time*. Commun. Math. Phys. **54** 1 (1977). Cited on page 5.

[38] R. M. Wald. *Trace anomaly of a conformally invariant quantum field in curved spacetime*. Phys. Rev. D **17** 1477 (1978). Cited on page 5.

[39] B. S. DeWitt. *Quantum Field Theory in Curved Space*. Physics Reports **19** 63 (1975). Cited on pages 5 and 38.

[40] S. M. Christensen. *Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background: The Covariant Point Separation Method*. Phys. Rev. **D14** 2490 (1976). Cited on pages 5 and 38.

[41] S. M. Christensen. *Regularization, Renormalization, and Covariant Geodesic Point Separation*. Phys. Rev. **D17** 946 (1978). Cited on pages 5 and 38.

[42] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press Cambridge, UK 1984. Cited on pages 5 and 38.

[43] D. Müller. *Homogeneous Solutions of Quadratic Gravity.* Int. J. Mod. Phys. Conf. Ser. **3** 111 (2011). arXiv:1107.3570. Cited on pages 5 and 38.

[44] A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko. *Vacuum quantum effects in strong fields.* Fridmann Lab. St. Petersburg 1994. Cited on pages 5 and 38.

[45] G. Gamow. *Expanding Universe and the Origin of Elements.* Physical Review **70** 572 (1946). Cited on page 6.

[46] E. Hubble and M. L. Humason. *The Velocity-Distance Relation among Extra-Galactic Nebulae.* Astrophys. J. **74** 43 (1931). Cited on page 6.

[47] G. Lemaître. *Republication of: A homogeneous universe of constant mass and increasing radius accounting for the radial velocity of extra-galactic nebulae.* General Relativity and Gravitation **45** 1635 (2013). Cited on page 6.

[48] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley and Sons New York 1972. Cited on pages 8, 39, 56, and 67.

[49] H. Stephani. *Relativity: An Introduction to Special and General Relativity.* Cambridge University Press (2004). Cited on pages 8, 39, 56, and 67.

[50] S. Carroll. *Spacetime and geometry: an introduction to general relativity; International ed.* Pearson Education Essex 2013. Cited on page 9.

[51] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger. *Theory of cosmological perturbations.* Physics Reports **215** 203 (1992). Cited on page 18.

[52] R. H. Brandenberger. *Lectures on the Theory of Cosmological Perturbations* pages 127–167. Springer Berlin Heidelberg Berlin, Heidelberg 2004. Cited on page 18.

[53] B. Jones. *Precision Cosmology: The First Half Million Years.* Cambridge University Press 2017. Cited on page 18.

[54] K. Array and et al. *BICEP2 / Keck Array VI: Improved Constraints On Cosmology and Foregrounds When Adding 95 GHz Data From Keck Array.* 2015). Cited on page 18.

[55] A. S. Goncharov and A. D. Linde. *Chaotic inflation in supergravity.* Physics Letters B **139** 27 (1984). Cited on page 19.

[56] T. Tenkanen and E. Tomberg. *Initial conditions for plateau inflation: a case study.* JCAP **04** 050 (2020). arXiv:2002.02420. Cited on page 19.

[57] S. Cotsakis. *Slice energy in higher-order gravity theories and conformal transformations.* Gravitation and Cosmology **14** 176–183 (2008). Cited on pages 20 and 21.

[58] A. De Felice and S. Tsujikawa. *f(R) Theories.* Living Reviews in Relativity **13** (2010). Cited on pages 20 and 22.

[59] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press 2011. Cited on page 20.

[60] K.-i. Maeda. *Towards the Einstein-Hilbert action via conformal transformation*. Phys. Rev. D **39** 3159 (1989). Cited on pages 21 and 60.

[61] T. V. Ruzmaikina and A. A. Ruzmaikin. *Quadratic Corrections to the Lagrangian Density of the Gravitational Field and the Singularity*. Soviet Journal of Experimental and Theoretical Physics **30** 372 (1969). Cited on pages 22 and 40.

[62] A. Kehagias, A. Moradinezhad Dizgah, and A. Riotto. *Remarks on the Starobinsky model of inflation and its descendants*. Physical Review D **89** (2014). Cited on page 22.

[63] M. V. Ostrogradsky. *Les Équations Différentielles Relatives au Problème des Isopérimètres*. Mem. Ac. St. Petersbourg **VI (4)** 385 (1850). Cited on pages 25, 26, and 27.

[64] A. Pais and G. E. Uhlenbeck. *On Field Theories with Non-Localized Action*. Phys. Rev. **79** 145 (1950). Cited on page 25.

[65] K. S. Stelle. *Classical Gravity with Higher Derivatives*. Gen. Rel. Grav. **9** 353 (1978). Cited on pages 25 and 39.

[66] R. Woodard. *Avoiding Dark Energy with 1/R Modifications of Gravity* pages 403–433. Springer Berlin Heidelberg Berlin, Heidelberg 2007. Cited on pages 25, 26, 27, 28, and 39.

[67] M. Trodden. *Theoretical Aspects of Cosmic Acceleration* 2016. `arXiv:1604.08899`. Cited on page 25.

[68] H.-J. Schmidt. *Stability and Hamiltonian formulation of higher derivative theories*. Phys. Rev. **D49** 6354 (1994). [Erratum: Phys. Rev.D54,7906(1996)]. `arXiv:gr-qc/9404038`. Cited on pages 25, 26, 27, and 28.

[69] W. P. F. de Medeiros. *Autoforça e a Técnica de Redução de Ordem*. Master's thesis Universidade de Brasíla 2017. Cited on page 26.

[70] K. R. Symon. *Mechanics*. Addison-Wesley Pub. Co Reading, Mass 3rd ed. edition 1971. Cited on pages 26 and 27.

[71] S. Thornton and J. Marion. *Classical Dynamics of Particles and Systems*. Brooks/Cole 2004. Cited on pages 26 and 27.

[72] L. Landau and E. Lifshitz. *Mechanics and Electrodynamics*. Elsevier Science 2013. Cited on pages 26 and 27.

[73] D. Griffiths. *Introduction to Electrodynamics*. Prentice Hall 1999. Cited on pages 29 and 31.

[74] J. Jackson. *Classical Electrodynamics, 3rd Edition*. John Wiley & Sons, Limited 1999. Cited on page 29.

[75] W. Rindler. *Kruskal Space and the Uniformly Accelerated Frame*. American Journal of Physics **34** 1174 (1966). Cited on page 33.

[76] E. Kreyszig. *Introductory functional analysis with applications*. Wiley classics library. Wiley New York; Chichester wiley classics library edition (1989). Cited on pages 38 and 65.

[77] A. S. Koshelev, L. Modesto, L. Rachwal, and A. A. Starobinsky. *Occurrence of exact R^2 inflation in non-local UV-complete gravity*. JHEP **11** 067 (2016). `arXiv:1604.03127`. Cited on page 40.

[78] W. Suen and P. Anderson. *Reheating in the Higher Derivative Inflationary Models*. Phys. Rev. D **35** 2940 (1987). Cited on page 41.

[79] S. Hassani. *Mathematical Methods: For Students of Physics and Related Fields*. Undergraduate Texts in Contemporary Physics. Springer New York (2000). Cited on page 42.

[80] P. Morse and H. Feshbach. *Methods of Theoretical Physics*. International series in pure and applied physics. McGraw-Hill (1953). Cited on page 42.

[81] C. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. Advanced Mathematical Methods for Scientists and Engineers. Springer 1999. Cited on page 55.