TOPOLOGICAL SOLITONS IN SCALAR FIELD THEORIES IN $(1+1)$-DIMENSIONS

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Dissertação apresentada ao Programa de Pós-Graduação em Física do Centro de Ciências Exatas da Universidade Federal do Espírito Santo como requisito parcial para a obtenção do grau de Mestre em Física, na área de concentração de Física Teórica

Orientador: Prof. Dr. Gabriel Luchini

Vitória

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"Soliton solutions of generalised BPS models in 1+1 dimensional space-time"

Tadeu Miguel Tassis

Dissertação submetida ao Programa de Pós-Graduação em Física da Universidade Federal do Espírito Santo como requisito parcial para a obtenção do título de Mestre em Física.
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"O binómio de Newton é tão belo como a Vénus de Milo.
O que há é pouca gente para dar por isso.
óóóó — óóóóóóóóó — óóóóóóóóóóóóóóó
(O vento lá fora)."
(Álvaro de Campos)

Resumo

Nessa dissertação nós estudamos uma generalização das equações BPS proposta recentemente, que pode ser aplicada a teorias com multipletos de campos escalares. Fazemos uma revisão geral do formalismo das equações BPS generalizadas e como este permite a construção de novos modelos a partir de álgebras de Lie, os modelos FKZ. Descrevemos os modelos baseados nas álgebras $\mathfrak{g}_2$ e $\mathfrak{su}(4)$ e apresentamos algumas soluções estáticas construídas a partir das equações BPS. Além disso, também apresentamos os resultados de algumas simulações feitas para o espalhamento de sólitons no modelo FKZ baseado na álgebra $\mathfrak{su}(2)$.

Palavras-chave: sólitons topológicos, equações BPS, simulação númerica.

Abstract

In this dissertation we study a generalization of the BPS equations, which can be applied in theories with multiplets of scalar fields. We give a overview of the generalized BPS formalism and how it can be applied to the construction of new models from Lie algebras, the FKZ models. We describe the models based on algebras $\mathfrak{g}_2$ and $\mathfrak{su}(4)$, and present some static solutions obtained from the BPS equations. Moreover, we also present the results of some simulations for soliton scattering processes in the FKZ model based on algebra $\mathfrak{su}(2)$.

Keywords: topological solitons, BPS equations, numerical simulation

Contents

Chapter 1

Introduction

In this dissertation we intend to present some aspects of the general study of topological solitons. Solitons arise in non-linear theories as localized configurations that are very stable. In some special kinds of theories, the so called integrable models [1, 2], the existence of an infinity of dynamically conserved quantities guarantees the stability of the configuration. Topological solitons can appear in theories which, in general, do not possess this infinity of conserved charges. They are stabilized by the topology of the configuration.

The general treatment of quantum field theory is to quantize the perturbations around a vaccum configuration of a given model and study the interaction of the particles that arise from this perturbation. It happens that some theories with non-trivial vacuum structure, possess non-trivial classical solutions called topological solitons [3,4] that interpolate distinct vacua. These solutions are localized and stable, thus the indiscriminate use of the word soliton in its broad sense. When we put two or more of these objects close to one another they even interact and have extremally complex scattering processes. And all of this is accomplished without the need of an extra particle carrying the interaction, like in usual quantum field theories.

The topological aspects of these solutions allow us to categorize and divide them in disjunct classes according to their topological characteristics. Topology, then, forbids configurations from one class to evolve dynamically into configurations of another. So if the solutions of a theory tend to minimize some functional, in our case, the energy functional, this means that, within a given class, the solutions with minimal energy are very stable, since they cannot decay into others dynamically.

Moreover, in some special theories, the energy has a lower bound – the Bogomolny bound – that is proportional to the number of solitons in a given configuration. The configurations that saturate this bound satisfy a simpler first order differential equation – the BPS equation –, which implies the usual second order Euler-Lagrange equations. This result is well known for theories with a single scalar field in $(1+1)$-d. Recently [5], though, this result was generalized for theories with a multiplet of fields. The formalism of generalized BPS equations, additionally, allows for the construction of a infinity of new models, each with their own pair of BPS equations and Bogomolny bounds. In [5] it was presented a group theoretical approach for the construction of a class of theories, the FKZ models.

Our main goals with this work is the review of the formalism of the generalized BPS equations, construction of the FKZ models based on algebras $\mathfrak{g}_2$ and $\mathfrak{su}(4)$ and numerical simulation of scattering processes in the simpler $\mathfrak{su}(2)$ FKZ models. In chapter 2 we will present – rather concisely – some ideas essential to later understand of the text. This includes some notions of topology, a derivation of Derrick's theorem for scalar theories and the introduction of the concept of BPS equations and Bogomolny bounds for theories with a single scalar field. In chapter 3 we will present a generalization of the idea of BPS equations for theories with multiplets of scalar fields and apply this formalism in the construction of the FKZ models. Chapter 4 will be regarding the numerical aspects of our work. We will present some simple numerical methods to integrate the BPS equations and to, later, simulate the scattering of solitons. Will also be presented here the results of some of the simulations we did with topological solitons of the $\mathfrak{su}(2)$ FKZ models.

Chapter 2

Mathematical background

This chapter is intended as a brief review of key concepts that will play a central role throughout this text. We begin by introducing some topological notions and how these abstract ideas are related to field theory. Then we take a look at Derrick's theorem and how it justifies our choice to work in one spatial dimension. Finally, we introduce the concept of BPS equations and Bogomolny bounds in the context of scalar field theories and present two canonical examples, namely the ϕ^4 and the sine-Gordon models.

2.1 Topology notions

The basic structure in topology [3,6,7] is the idea of topological spaces, which serves as a foundation for the entire field.

Definition 2.1. A **topological space** is a set X together with a collection S of subsets of X, called the **open sets** of X, satisfying

(i) the empty set and X itself are open,

(ii) given any number of open sets $U_\alpha \subseteq X$, their union $\bigcup_\alpha U_\alpha$ is open,

(iii) given any **finite** number of open sets $U_\alpha \subseteq X$, their intersection $\bigcap_\alpha U_\alpha$ is open.

The collection of open sets of X is called the **topology** of X. Other important concepts are those of continuous maps and homeomorphisms between topological spaces.

Definition 2.2. A map $f : X \to Y$ between topological spaces is called a **continuous map** if, given any open set $U \subseteq Y$, the inverse image $f^{-1}\{U\} \subseteq X$ is open.

Definition 2.3. A map $f : X \to Y$ is called a **homeomorphism** if it is both continuous and has a continuous inverse.

If there is such a homeomorphism between X and Y, we say that X is homeomorphic to Y. Moreover, the homeomorphisms between topological spaces define an **equivalence relation**:

(i) X is clearly homeomorphic to itself as we can always define an identity map $id : X \to X$ by $id(x) = x$ *(reflexive)*

(ii) Since f has a continuous inverse, we have that if X is homeomorphic to Y, Y is also homeomorphic to X *(symmetric)*

(iii) If X is homeomorphic to Y and Y is homeomorphic to Z, it can be shown by composition of maps that X is homeomorphic to Z *(transitive)*

That is, we can divide a collection of topological spaces into disjoint **equivalence classes**. Being in the same equivalence class of X, which we write $[X]$, means that any space $\tilde{X} \in [X]$ can be continuously deformed into X and, in that respect, we can categorize which spaces are topologicaly equivalent and which are not. Moreover, a central subject of topology is the study of properties of a given space that do not change when we deform it, the **topological invariants**.

Somewhat more important for our specific needs is the idea of homotopy.

Definition 2.4. Given two continuous maps $f : X \to Y$ and $g : X \to Y$, f is said to be **homotopic** to g if there is a continuous map $\psi : X \times [0, 1] \to Y$ satisfying

$$\psi(x, 0) = f(x) \tag{2.1}$$

$$\psi(x, 1) = g(x) \tag{2.2}$$

The fact that f is continuous means that, as the parameter t changes from 0 to 1, f is continuously deformed into g, and *vice versa*, since we also look at variations from 1 to 0, thus being homotopic is a *symmetric* relation. In fact, it is easy to show *reflexivity* and *transitivity* of the relation. Hence, homotopy lets us separate the space of continuous maps from X to Y, which we write $C(X, Y)$, in equivalence classes known as the **homotopy classes**. Because homeomorphisms are continuous maps, the homotopy classes remain unchanged when we look at spaces homeomorphic to both X or Y, *i.e.*, homotopy is a topological invariant of the pair of spaces. One interesting thing we can do next is to fix the domain space X to be a specific one, usualy the n-sphere S^n. This way we can study the homotopy classes of $C(S^n, Y)$ and how they change as we choose different target spaces Y. In fact, the set of homotopy classes of $C(S^n, Y)$, denoted $\pi_n(Y)$, has a group structure and is called the **n-th homotopy group** of Y.

The most important homotopy group for our study is actualy the simpler one, $\pi_0(Y)$, which is the group of homotopy classes of maps from the 0-sphere to space Y [1]. Another important example is the first homotopy group $\pi_1(Y)$, also called the **fundamental group** of Y. A map from the circle S^1 defines a closed path on target space Y, which we call a **loop**, so the elements of the fundamental group are the equivalence classes of loops that can be continuously deformed into each other. Intuitively, one can think that for a path to not be

[1] The 0-sphere is defined here as the closure of a closed interval $[a, b] \subset \mathbb{R}$, *i.e.*, the set of any two non-coincident points: $\{a, b\} \subset \mathbb{R}$, such that $a \neq b$.

able to be deformed into another there should be something in space Y standing in its way, like a hole, for example. And is exactly this type of information about the shape of Y that the homotopy groups give. The subject of topology is a vast one and the above only scratches its surface, but it should give us enough tools to understand the topological character of the solutions we want to study.

2.2 Topological considerations in field theories

Our main goal is to study relativistic scalar theories defined on $(d+1)$ flat spacetime. As we will see, the topological information contained in these fields lie on the values they assume at spatial infinity [3] and, as time evolution can be thought of as a continuous variation of the fields, it should not change their topological character. At this stage, it is enough, then, to consider only static configurations. Further, it is always possible, after solving the equations in the reference frame where the solutions are static, to make a Lorentz boost and get a dynamic solution.

Let us consider a multiplet of n real scalar fields defined on d-dimensional euclidean space

$$\phi : \mathbb{R}^d \to \mathbb{R}^n. \tag{2.3}$$

The static energy functional is given by

$$E = \int d^d x \left[\frac{1}{2}\partial_i\phi^a\partial_i\phi^a + U(\phi)\right] \tag{2.4}$$

where U, the potential, is a non-negative function of the fields. Since both the kinetic and potential terms are non-negative by definition, the energy functional is also non-negative. The physical field configurations we are interested in have finite energy and are defined as those which make the functional stationary against small variations, which is analogous to saying that the physical field configurations are solutions of the following set of second order equations

$$\partial_i\partial_i\phi^a - \frac{\partial U}{\partial\phi^a} = 0 \tag{2.5}$$

that also have finite energy.

As the potential is a non-negative function, it will have global minima. We can always take these minima to be $U_{min} = 0$ by addition of a suitable constant. Let us define then the vacuum manifold $\mathcal{V}$ as the set of points of field space that make the potential null, *i.e.*

$$\mathcal{V} = \{\phi_0; U(\phi_0) = 0\} \tag{2.6}$$

In particular, a **vacuum configuration** is one for which $\phi(x) = \phi_0\ \forall\ x$ and $\phi_0 \in \mathcal{V}$. The physical requirement that the energy should be finite imposes that the energy density must be localized, otherwise the integral would not converge. Rigorously, this means that the energy density ε

$$\varepsilon = \frac{1}{2}\partial_i\phi^a\partial_i\phi^a + U(\phi) \tag{2.7}$$

should be such that $\varepsilon \to 0$ as $r \to \infty$, where r is the distance from the origin of space coordinates. As both kinetic and potential terms are non-negative, this implies they both tend to zero as well, which imposes boundary conditions on the possible field configurations:

$$\phi \to \phi_0 \; ; \phi_0 \in \mathcal{V} \tag{2.8}$$

$$\partial_i\phi \to 0 \tag{2.9}$$

as $r \to \infty$. The boundary condition for the field, ϕ^∞, may be thought of as a mapping from the $(d-1)$-sphere at spatial infinity to the vacuum manifold, *i.e.*,

$$\phi^\infty : S^{d-1}_\infty \to \mathcal{V} \tag{2.10}$$

where $S^{d-1}_\infty \subset \mathbb{R}^d$ denotes the sphere with radius $r \to \infty$. Moreover, even if two field configurations ϕ and $\widetilde{\phi}$ have different asymptotic data ϕ^∞ and $\widetilde{\phi}^\infty$, they still can be continuously deformed into one another if ϕ^∞ and $\widetilde{\phi}^\infty$ are homotopic. Which means that the topological character of a given field configuration ϕ lies entirely in the homotopy class of its asymptotic data $[\phi^\infty] \in \pi_{d-1}(\mathcal{V})$.

The important thing to notice is that if a configuration is in a specific homotopy class $[\phi^\infty]$, it cannot be taken into another by continuous deformations. In paricular, time evolution under a differential equation is a continuous deformation of the field. So if a theory has non-trivial homotopy group $\pi_{d-1}(\mathcal{V})$ and a particular configuration ϕ is in a different homotopy class $[\phi^\infty]$ than the homotopy class of a vaccum configuration $[\phi_0^\infty]$, then the configuration ϕ cannot decay in a vacuum configuration by continuous time evolution. This gives a topological stability to non-trivial solutions.

In the next section we will be given arguments that show the theories we are mainly interested in can only occur in $d = 1$ spatial dimension. So we will only have to deal with the simplest type of homotopy group, $\pi_0(\mathcal{V})$, which consists of the pairs $\{\phi_-, \phi_+\}$, where $\phi_\pm = \lim_{x\to\pm\infty}\phi$. The configurations belonging to the non-trivial homotopy classes, *i.e.*, those that are not in the vaccum class[2], are called **topological solitons**. The terminology soliton is an alusion to the finite energy solutions of integrable models, although some might find the use of the same terminology to be bit of a stretch. Moreover, in the case of integrable models, these non-trivial solutions are stabilized locally by an infinity of conserved charges. Here, they will be stable due to topology, since it prevents these objects to decay in a vaccum configuration by time evolution.

[2]This is regarding 1-soliton solutions. It is possible to have multi-soliton solutions in the same class of the vaccum. Although in this case there is no topological constraint to avoid these configurations to decay in a vacuum.

2.3 Derrick's theorem

Derrick's theorem [3,4] is actually a statement on the non-existence of static field configurations with finite energy for theories defined on larger numbers of spatial dimensions. To prove it we use the fact that a physical field configuration should make the energy functional stationary against small variations. One such variation is a rescale of space coordinates $x \mapsto \lambda x$. This defines a family of fields[3]

$$\phi(\lambda, x) \equiv \phi(\lambda x) \tag{2.11}$$

and the energy functional becomes a function of the parameter λ

$$E(\lambda) = E[\phi(\lambda, x)]. \tag{2.12}$$

If $\phi(1, x) = \phi(x)$ is a solution of the field equations, this means that $\lambda = 1$ should be a stationary point of the energy, *i.e.*,

$$\left.\frac{dE}{d\lambda}\right|_{\lambda=1} = 0 \tag{2.13}$$

So, for the one-parameter family of fields, the energy functional becomes rewritten in terms of the rescaled coordinate $y = \lambda x$, with

$$\begin{cases} d^d x = \lambda^{-d} d^d y \\ \partial/\partial x^i = \lambda \partial/\partial y^i, \end{cases} \tag{2.14}$$

as

$$\begin{align} E(\lambda) &= \int d^d y\ \lambda^{-d}\left[\lambda^2 \frac{1}{2}\partial_i\phi^a(y)\partial_i\phi^a(y) + U(\phi(y))\right] \tag{2.15}\\ &= \lambda^{2-d} T + \lambda^{-d} V \tag{2.16} \end{align}$$

where $T = \int d^d x\ \frac{1}{2}\partial_i\phi^a\partial_i\phi^a$ and $V = \int d^d y\ U$ are the kinetic and potential energies, respectively. Because $\lambda = 1$ is a stationary point, we have

$$\begin{align} \frac{dE}{d\lambda} &= (2-\lambda)\lambda^{1-d} T - d\lambda^{-1-d} V \nonumber\\ \Rightarrow \left.\frac{dE}{d\lambda}\right|_{\lambda=1} &= (2-d)T - dV = 0. \tag{2.17} \end{align}$$

Since $T, V \geq 0$, the equality leads to an absurd for $d > 2$, proving that we cannot have static solutions with finite energy in these types of theories. For $d = 2$, equation (2.17) implies that $V = 0$, *i.e.*, we can have static solutions here provided that the potential is null. For $d = 1$, we can have static solutions and (2.17) implies the virial theorem $T = V$.

[3]Which are here resctricted to be scalar fields, but the general argument remains the same for other types of fields.

Derrick's theorem, then, only allows for the existence of scalar theories with finite energy non-trivial solutions for $d = 1$ and $d = 2$. The case $d = 2$ comes with the extra condition that the potential term should be null, nevertheless this is exactly the case for theories like the non-linear sigma model, which has lump solutions. Moreover, there are other ways to avoid the results of this theorem as well and have stable static solutions in higher dimensions. If we add a gauge field in the mix, for example, we can have vortices in $d = 2$ and monopoles in $d = 3$. Or if we consider theories with higher order derivatives (or higher powers of these), we have baby-skyrmions and skyrmions in $d = 2$ and $d = 3$, respectively [3, 4]. But since our interest is to study scalar theories with energy functional of the type (2.4), we do not have much choice other than consider models defined on one spatial dimension.

2.4 BPS equations and the Bogomolny bound

In general, the theories we are used to deal with have solutions which satisfy sets of second order differential equations. However it was shown independently by Bogomolny [8] and by Prasad and Sommerfield [9] that some theories, or certain limits of these theories, have solutions that satisfy a simpler set of first order equations, the **BPS equations**. Further, it is possible to show that the energy satisfies an inequality, the **Bogomolny bound**, and is bounded by values that depend on the topological sector the solutions are in, with the minimal value being reached for solutions of the BPS equations.

Let us construct this inequality for the simple case of a single scalar field theory which is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi). \tag{2.18}$$

As a consequence of the stationary action principle, any solution to this theory has to satisfy the Euler-Lagrange equation

$$\partial_\mu\partial^\mu\phi + \frac{dU}{d\phi} = 0. \tag{2.19}$$

The static energy functional of this theory is given by

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + U(\phi)\right]. \tag{2.20}$$

Now, this functional can be rewritten as

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\left(\frac{d\phi}{dx} \mp \sqrt{2U}\right)^2 \pm 2\frac{d\phi}{dx}\sqrt{2U}\right] \tag{2.21}$$

and noticing that the first term is non-negative, the following relation holds true:

$$\begin{aligned} E &\geq \pm \int_{-\infty}^{\infty} dx \, \frac{d\phi}{dx} \sqrt{2U} \\ &\geq \left| \int_{-\infty}^{\infty} dx \, \frac{d\phi}{dx} \sqrt{2U} \right| \\ \Rightarrow \;\; E &\geq \left| \int_{\phi_-}^{\phi_+} d\phi \;\; \sqrt{2U} \right| \end{aligned} \tag{2.22}$$

where $\phi_\pm = \lim_{x\to\pm\infty} \phi(x)$.

Equation (2.22) gives the so called Bogomolny bound. Equality in (2.21) holds when the field satisfies

$$\frac{d\phi}{dx} = \pm \sqrt{2U}, \tag{2.23}$$

a simpler first order equation, known as the BPS equation. If we take a spatial derivative of (2.23), then

$$\frac{d}{dx}\left(\frac{d\phi}{dx} \mp \sqrt{2U}\right) = \frac{d^2\phi}{dx^2} \mp \frac{1}{\sqrt{2U}} \frac{dU}{dx} \tag{2.24}$$

$$= \frac{d^2\phi}{dx^2} \mp \frac{1}{\sqrt{2U}} \frac{d\phi}{dx} \frac{dU}{d\phi} \, , \quad \frac{d\phi}{dx} = \pm \sqrt{2U} \tag{2.25}$$

$$= \frac{d^2\phi}{dx^2} - \frac{dU}{d\phi} = 0, \tag{2.26}$$

and we see that a static configuration that is a solution to the first order BPS equations is also a solution to the second order E-L equation (2.19). In order to accentuate the topological character of the Bogomolny bound, we can define the prepotential W to be a function satisfying

$$\frac{dW}{d\phi} \equiv \sqrt{2U} \tag{2.27}$$

Substituting this definition into (2.22)

$$\begin{aligned} E &\geq \left| \int_{\phi_-}^{\phi_+} d\phi \, \frac{dW}{d\phi} \right| \\ &\geq \left| W(\phi_+) - W(\phi_-) \right| . \end{aligned} \tag{2.28}$$

That is, the bound on the energy only depends on the boundary conditions of the field. Now we present two well known examples of theories described by a single real scalar field, known as the ϕ^4 and sine-Gordon models.

2.4.1 The ϕ^4 model

The ϕ^4 model [3,4] is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\left(\phi^2 - 1\right)^2 \tag{2.29}$$

thus its static energy is

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{4}\left(\phi^2 - 1\right)^2\right], \tag{2.30}$$

which gives us the static field equation

$$\frac{d^2\phi}{dx^2} - \phi\left(\phi^2 - 1\right) = 0. \tag{2.31}$$

The potential $U = \frac{1}{4}(\phi^2 - 1)^2$ is shown in figure 2.1. It is zero for $\phi = \pm 1$, thus the vacuum manifold is simply

$$\mathcal{V} = \{-1, 1\} \cong \mathbb{Z}_2. \tag{2.32}$$

This theory, then, has non-trivial homotopy group $\pi_0(\mathcal{V})$

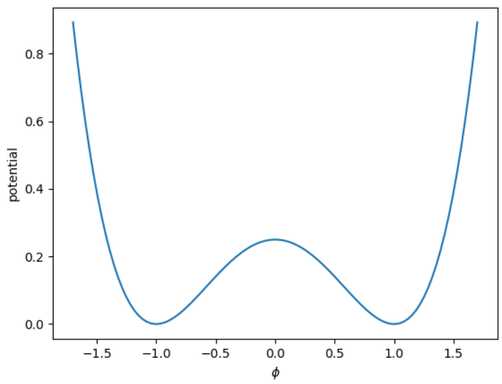


Figure 2.1: Potential $U(\phi)$ of the ϕ^4 model

$$\pi_0(\mathcal{V}) = \{\{-1, -1\}, \{-1, 1\}, \{1, -1\}, \{1, 1\}\} \tag{2.33}$$

which allows for the existence of topological solitons.

The topological degree N defined by

$$N \equiv \frac{1}{2}\int_{-\infty}^{\infty} dx\, \frac{d\phi}{dx} \tag{2.34}$$
$$= \frac{1}{2}\int_{\phi_-}^{\phi_+} d\phi \tag{2.35}$$
$$= \frac{\phi_+ - \phi_-}{2} \tag{2.36}$$

summarizes all of the topological data of a given configuration, thus it should be invariant under time evolution. Its possible values are $N \in \{-1, 0, 1\}$, with $N = 0$ being the degree of solutions in the homotopy classes of the vacua configurations $[\{-1, -1\}]$ and $[\{-1, -1\}]$, *i.e.*, solutions that can eventually decay in a vacuum configuration under time evolution. The cases $N = 1$ and $N = -1$ are the degrees of solutions in the classes $[\{-1, 1\}]$ and $[\{1, -1\}]$, respectively. In these cases the field cannot decay into a vaccum configuration. The Bogomolny bound (2.22) for this theory is

$$E \geq \left| \int_{\phi_-}^{\phi_+} d\phi\ \sqrt{2U} \right| \tag{2.37}$$
$$\geq \left| \int_{\phi_-}^{\phi_+} d\phi\ \sqrt{\frac{1}{2}(\phi^2 - 1)^2} \right| \tag{2.38}$$
$$\geq \frac{1}{\sqrt{2}} \left| \int_{\phi_-}^{\phi_+} d\phi\ |\phi^2 - 1| \right|. \tag{2.39}$$

Since the integration stays in the range $|\phi| \leq 1$, the subtraction inside the inner modulus will always be negative

$$E \geq \frac{1}{\sqrt{2}} \left| \int_{\phi_-}^{\phi_+} d\phi\ \left(1 - \phi^2\right) \right| \tag{2.40}$$
$$\geq \frac{1}{\sqrt{2}} \left| \left[\phi - \frac{\phi^3}{3} \right]_{\phi_-}^{\phi_+} \right| \tag{2.41}$$

and because for the possible boundary conditions $\phi_\pm^3 = \phi_\pm$ we have

$$E \geq \frac{1}{\sqrt{2}} \left| \left[\frac{2\phi}{3} \right]_{\phi_-}^{\phi_+} \right| \tag{2.42}$$
$$\geq \frac{\sqrt{2}}{3} \left| \left(\phi_+ - \phi_-\right) \right|, \tag{2.43}$$

the Bogomolny bound for the ϕ^4 model is then finally defined by

$$E \geq \frac{2\sqrt{2}}{3}|N|. \tag{2.44}$$

The equality in the above relation holds for the solutions of the BPS equations (2.23) which in this case read

$$\frac{d\phi}{dx} = \pm\frac{1}{\sqrt{2}}\left(1 - \phi^2\right). \tag{2.45}$$

These equations can be easily integrated if one considers the substitution

$$\phi = \tanh(\psi), \tag{2.46}$$

for which it becomes, using that $1 - \phi^2 = \operatorname{sech}^2\psi$ and $\frac{d\phi}{dx} = \operatorname{sech}^2\psi\frac{d\psi}{dx}$,

$$\frac{d\psi}{dx} = \pm\frac{1}{\sqrt{2}} \tag{2.47}$$

which has solution

$$\psi = \pm\frac{x - x_0}{\sqrt{2}} \tag{2.48}$$

where x_0 is an integration constant. Further, plugging this result back into the definition of ϕ in terms of ψ, the solution of the BPS equation is finally given[4]:

$$\phi = \pm\tanh\left(\frac{x - x_0}{\sqrt{2}}\right). \tag{2.49}$$

Expressions (2.49) give the **kink** (positive sign) and **antikink** (negative sign) solutions of the ϕ^4 model, with topological degrees $N = 1$ and $N = -1$, respectively. In figures 2.2a and 2.2b we have the kink and antikink solutions plotted, as well as their energy densities. Notice that the energy density is localized, which guarantees finite energy. Being solutions of the BPS equations, they saturate the Bogomolny bound and both have static energy

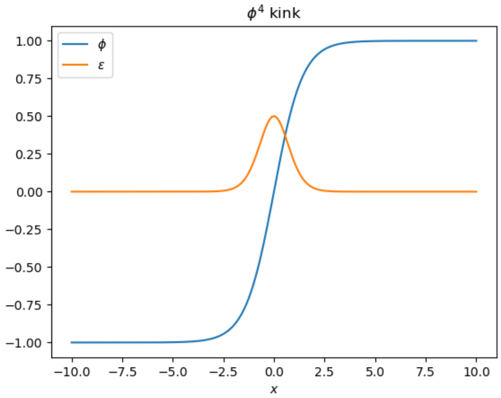
$$E = \frac{2\sqrt{2}}{3}. \tag{2.50}$$

2.4.2 The sine-Gordon model

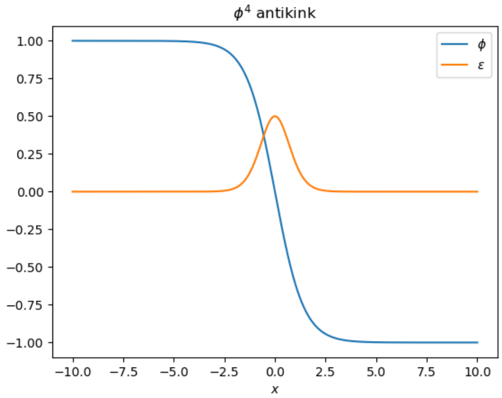
The sine-Gordon model [3, 4] is defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \left(1 - \cos\phi\right) \tag{2.51}$$

[4]Where we have used the fact that the tanh is an odd function.



(a) The kink solution and its energy density.



(b) The anti-kink solution and its energy density

Figure 2.2: The kink and anti-kink are solutions of the ϕ^4 model characterised by their topological degree of $+1$ and -1 respectively. These are localised finite energy solutions.

and have static energy

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + \left(1 - \cos\phi \right) \right]. \tag{2.52}$$

The static E-L equation reads

$$\frac{d^2\phi}{dx^2} - \sin\phi = 0. \tag{2.53}$$

The potential $U(\phi) = 1 - cos\phi$ is shown in figure 2.3. The vaccum manifold is composed by the points of field space for which $\cos\phi = 1$, *i.e.*,

$$\mathcal{V} = \{2\pi n; n \in \mathbb{Z}\} \cong \mathbb{Z} \tag{2.54}$$

Since we have the symmetry $\phi \mapsto \phi + 2\pi$, we can take $\phi_- = 0$ wihout loss of generality.

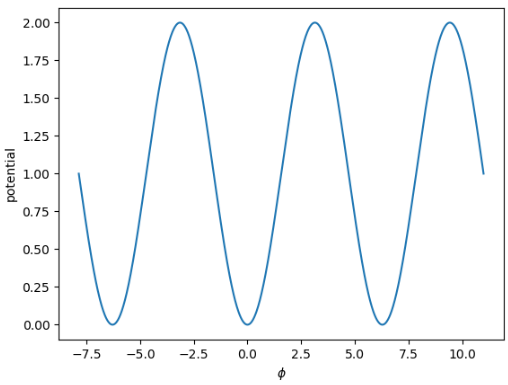


Figure 2.3: Potential $U(\phi)$ of the sine-Gordon model

Thus, the homotopy group of this theory is

$$\pi_0(\mathcal{V}) = \{\{0, 2\pi n\}; n \in \mathbb{Z}\} \cong \mathbb{Z}. \tag{2.55}$$

The topological degree of possible solutions is therefore

$$N = \frac{1}{2\pi} \int_{\phi_-}^{\phi_+} d\phi = \frac{1}{2\pi} \int_0^{2\pi n} d\phi = n \tag{2.56}$$

and the Bogomolny bound (2.22) is given by

$$\begin{aligned} E \geq \left| \int_0^{2\pi N} d\phi \, \sqrt{2(1-\cos\phi)} \right| &= 2 \left| \int_0^{2\pi N} d\phi \left| \sin\frac{\phi}{2} \right| \right| & (2.57) \\ &= 2|N| \left| \int_0^{2\pi} d\phi \left| \sin\frac{\phi}{2} \right| \right| & (2.58) \\ &= 2|N| \left| \left[-2\cos\frac{\phi}{2} \right]_0^{2\pi} \right|, & (2.59) \end{aligned}$$

where we have used the fact that $|\sin(\phi/2)|$ is periodic and that $\sin(\phi/2)$ is always positive in the range $[0, 2\pi]$. Thus,

$$E \geq 8|N|. \tag{2.60}$$

The equality will hold for the solutions of the BPS equation

$$\frac{d\phi}{dx} = \pm 2 \sin\frac{\phi}{2}. \tag{2.61}$$

The integration of that equation can be easily done using the substitution $\psi = \log\left(\tan\frac{\phi}{4}\right)$, $\frac{d\psi}{dx} = \frac{1}{2\sin\frac{\phi}{2}} \frac{d\phi}{dx}$, which results in the simple equation

$$\frac{d\psi}{dx} = \pm 1 \tag{2.62}$$

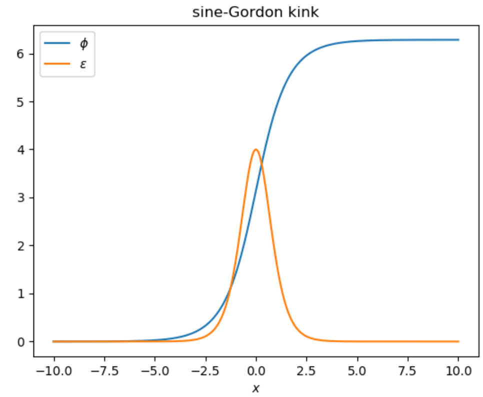
for which the solution is

$$\Rightarrow \;\; \psi = \pm(x - x_0),$$

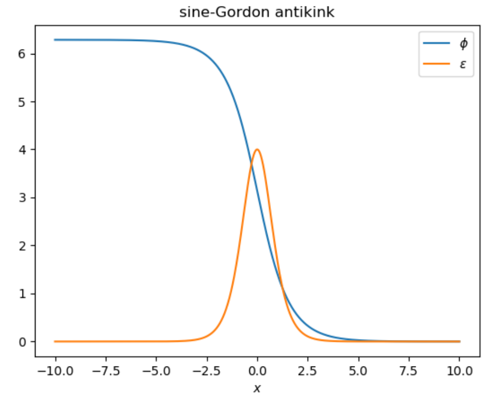
where x_0 is an integration constant. Inverting this relation, $\phi = 4\arctan(e^{\psi})$, the solution of the sine-Gordon equation reads

$$\phi = 4\arctan\left(e^{\pm(x-x_0)}\right). \tag{2.63}$$

For the positive sign, this gives a kink interpolating 0 and 2π, and the negative sign gives a antikink interpolating 2π and 0. Using the symmetry $\phi \mapsto \phi + 2\pi$, we can get a kink of antikink interpolating any two consecutive vacua. In figures 2.4a and 2.4b we have the plots



(a) The kink solution and its energy density.



(b) The anti-kink solution and its energy density

Figure 2.4: The kink and anti-kink are solutions of the sine-Gordon model characterised by their topological degree of $+1$ and -1 respectively. These are localised finite energy solutions.

of the solution (2.63).

Solutions which interpolate non-consecutive vacua are not BPS states, *i.e.*, do not satisfy the BPS equations. These solutions will satisfy a strict Bogomolny bound

$$E > 8|N|. \tag{2.64}$$

In particular, it is possible to construct these solutions analytically, due to the fact that the sine-Gordon model is integrable [1,2], that is, besides topology, it has an infinity of conserved charges that stabilize its configurations. The solutions interpolating non-consecutive vacua are called multi-soliton solutions, as they can be seen as a sum of individual solitons when this objects are asymptotically apart. Moreover these configurations are usually time dependent solutions that describe the movement of the individual solitons and their scattering. The strict Bogomolny bound that they satisfy accounts, then, for the energies of the individual solitons and their interaction. The models we will study here do not seem, in general, to be integrable, yet it is possible to study the scattering of the configurations numerically, as we will show in chapter 4.

Chapter 3

Generalized BPS equations and FKZ models

In the last chapter we presented a fairly general way to determine the BPS equations and the Bogomolny bound of theories with a single scalar field. Now we take a look at a generalization [5] of the method which allows us to define those same concepts for theories of multiplets of scalar fields.

3.1 Generalized BPS equations

Let us have a multiplet of n scalar fields ϕ_a defined on one spatial dimension

$$\phi : \mathbb{R} \to \mathbb{R}^n. \tag{3.1}$$

Consider a generic static energy functional of the form

$$E \equiv \frac{1}{2} \int_{-\infty}^{\infty} dx \, \left[A_\alpha^2 + \widetilde{A}_\alpha^2 \right] \tag{3.2}$$

where A and $\widetilde{A}$ are functions of only the fields ϕ_a and their spatial derivatives ϕ_a', and α stands for any indices there might have. A static solution, then, should be a stationary point of this functional with respect to a variation throught $\delta\phi_a$

$$\begin{aligned}
\delta E &= \int dx \, \left[A_\alpha \left(\frac{\partial A_\alpha}{\partial \phi_a} \delta\phi_a + \frac{\partial A_\alpha}{\partial \phi_a'} \delta\phi_a' \right) + A \leftrightarrow \widetilde{A} \right] \\
&= \int dx \, \left[A_\alpha \frac{\partial A_\alpha}{\partial \phi_a} \delta\phi_a - \frac{d}{dx} \left(A_\alpha \frac{\partial A_\alpha}{\partial \phi_a'} \right) \delta\phi_a + \frac{d}{dx} \left(A_\alpha \frac{\partial A_\alpha}{\partial \phi_a'} \delta\phi_a \right) + A \leftrightarrow \widetilde{A} \right] \\
&= \int dx \, \left[A_\alpha \frac{\partial A_\alpha}{\partial \phi_a} - \frac{d}{dx} \left(A_\alpha \frac{\partial A_\alpha}{\partial \phi_a'} \right) + A \leftrightarrow \widetilde{A} \right] \delta\phi_a \qquad (3.3)
\end{aligned}$$

$$\tag{3.4}$$

where $A \leftrightarrow \widetilde{A}$ means we repeat the terms but exchange A and $\widetilde{A}$, and where we have used the fact that $\delta\phi_a = 0$ at $x = \pm\infty$ in order to make the terms with total derivatives in the integration to vanish. If the first order variation of E with respect to ϕ is supposed to leave E stationary, *i.e.*, $\delta E = 0$, then ϕ should satisfy the second order equations

$$\frac{d}{dx}\left(A_\alpha \frac{\partial A_\alpha}{\partial \phi'_a} + \widetilde{A}_\alpha \frac{\partial \widetilde{A}_\alpha}{\partial \phi'_a}\right) - A_\alpha \frac{\partial A_\alpha}{\partial \phi_a} - \widetilde{A}_\alpha \frac{\partial \widetilde{A}_\alpha}{\partial \phi_a} = 0 \tag{3.5}$$

which we indentify as the Euler-Lagrange equations for this theory. We also consider a quantity that we call the topological charge Q defined as the functional

$$Q \equiv \int_{-\infty}^{\infty} dx\, A_\alpha \widetilde{A}_\alpha \tag{3.6}$$

which must be left stationary[1] against variations of ϕ

$$\begin{aligned}
\delta Q &= \int dx\, \left[\delta A_\alpha \widetilde{A}_\alpha + A \leftrightarrow \widetilde{A}\right] \\
&= \int dx\, \left[\left(\frac{\partial A_\alpha}{\partial \phi_a}\delta\phi_a + \frac{\partial A_\alpha}{\partial \phi'_a}\delta\phi'_a\right)\widetilde{A}_\alpha + A \leftrightarrow \widetilde{A}\right] \\
&= \int dx\, \left[\frac{\partial A_\alpha}{\partial \phi_a}\widetilde{A}_\alpha\delta\phi_a - \frac{d}{dx}\left(\frac{\partial A_\alpha}{\partial \phi'_a}\widetilde{A}_\alpha\right)\delta\phi_a + \frac{d}{dx}\left(\frac{\partial A_\alpha}{\partial \phi'_a}\widetilde{A}_\alpha\delta\phi_a\right) + A \leftrightarrow \widetilde{A}\right] \\
&= \int dx\, \left[\frac{\partial A_\alpha}{\partial \phi_a}\widetilde{A}_\alpha - \frac{d}{dx}\left(\frac{\partial A_\alpha}{\partial \phi'_a}\widetilde{A}_\alpha\right) + A \leftrightarrow \widetilde{A}\right]\delta\phi_a \qquad (3.7) \\
& \qquad (3.8)
\end{aligned}$$

where we have used the fact that $\delta\phi_a = 0$ at $x = \pm\infty$. Stationarity condition implies that ϕ should satisfy

$$\frac{d}{dx}\left(\frac{\partial A_\alpha}{\partial \phi'_a}\tilde{A}_\alpha + A_\alpha \frac{\partial \tilde{A}_\alpha}{\partial \phi'_a}\right) - \frac{\partial A_\alpha}{\partial \phi_a}\tilde{A}_\alpha - A_\alpha \frac{\partial \tilde{A}_\alpha}{\partial \phi_a} = 0 \tag{3.9}$$

Further, if we impose the set of first order equations

$$A_\alpha = \pm\tilde{A}_\alpha \tag{3.10}$$

then we see that equations (3.5) and (3.9) become equivalent. Moreover, the energy functional can be written as

$$E = \frac{1}{2}\int dx\, \left[A_\alpha \mp \tilde{A}_\alpha\right]^2 \pm \int dx\, A_\alpha \tilde{A}_\alpha \tag{3.11}$$

[1] This condition is crucial if Q is to have the meaning of a topological degree: it must depend only on the data at the boundary, being unchanged under deformations of the field which preserve that data.

and because the first term is non-negative and noticing that the second one is exactly our definition of topological charge, we find

$$E \geq |Q|. \tag{3.12}$$

We identify equation (3.12) as the generalized version of the Bogomolny bound. Since the equality holds for solutions of (3.10), we indentify these as the generalized BPS equations.

3.1.1 Generalized BPS equations for multiplets of scalar fields

In order to fit this formalism in the description of scalar theories, we define the objetcs A and $\widetilde{A}$ to be

$$\begin{aligned} A_\alpha &\equiv k_{ab}\frac{d\phi_b}{dx} \\ \tilde{A}_\alpha &\equiv \frac{\partial W}{\partial \phi_b}k^{-1}_{ba} \end{aligned} \tag{3.13}$$

where W is a function of the fields and is called the prepotential, and k is an invertible matrix. So far, being invertible is the only restriction we have to make about the matrix k. Further, its entries could depend on ϕ, ϕ' or even other fields and their derivatives. For the sake of simplicity we consider, from now on, that it depends only on the fields ϕ alone. Thus, for these theories we have the BPS equations (3.10)

$$\begin{aligned} k_{ab}\frac{d\phi_b}{dx} &= \pm\frac{\partial W}{\partial \phi_b}k^{-1}_{ba} \\ \therefore\ \eta_{ab}\frac{d\phi_b}{dx} &= \pm\frac{\partial W}{\partial \phi_a} \end{aligned} \tag{3.14}$$

where $\eta \equiv k^T k$ is an invertible symmetric matrix. Multiplying the inverse of η gives us the BPS equations

$$\frac{d\phi_a}{dx} = \pm\eta^{-1}_{ab}\frac{\partial W}{\partial \phi_b}. \tag{3.15}$$

We can write the energy functional for such a theory as

$$\begin{aligned} E &= \frac{1}{2}\int dx\left[k_{ca}k_{cb}\frac{d\phi_a}{dx}\frac{d\phi_b}{dx} + \frac{\partial W}{\partial \phi_a}\frac{\partial W}{\partial \phi_b}k^{-1}_{ac}k^{-1}_{bc}\right] \\ &= \frac{1}{2}\int dx\left[\eta_{ab}\frac{d\phi_a}{dx}\frac{d\phi_b}{dx} + \eta^{-1}_{ab}\frac{\partial W}{\partial \phi_a}\frac{\partial W}{\partial \phi_b}\right] \\ &= \int dx\left[\frac{1}{2}\eta_{ab}\frac{d\phi_a}{dx}\frac{d\phi_b}{dx} + U\right] \end{aligned} \tag{3.16}$$

where we have defined the potential

$$U \equiv \frac{1}{2}\eta^{-1}_{ab}\frac{\partial W}{\partial \phi_a}\frac{\partial W}{\partial \phi_b}. \tag{3.17}$$

Since we want the energy to be bounded from below, it suffices to impose that η is a positive definite matrix and that U is a non-negative function. It is easy to note that the extrema of W, that is, the points ϕ_0 such that

$$\left.\frac{\partial W}{\partial \phi_a}\right|_{\phi=\phi_0} = 0 \tag{3.18}$$

are vacua of these theories, *i.e*, the values of ϕ for which $U(\phi) = 0$. In fact, it can be shown [5] that those are the only vacua of these theories. That is, the minima $U(\phi)$ are always critical points of the prepotential $W(\phi)$.

We see that the topological charge of these theories will be of the form

$$\begin{aligned} Q &= \int dx\, k_{ab}\frac{d\phi_b}{dx}\frac{\partial W}{\partial \phi_c}k^{-1}_{ca} \\ &= \int dx\, \frac{d\phi_a}{dx}\frac{\partial W}{\partial \phi_a} \\ &= \int_\gamma d\phi \cdot \nabla_\phi W \end{aligned} \tag{3.19}$$

which is a line integral of the gradient – with respect to the fields – of W over the path γ in field space. By the fundamental theorem of calculus, this integral is simply the function W evaluated at the endpoints of the path, $\phi_\pm$, that is,

$$Q = W(\phi_+) - W(\phi_-). \tag{3.20}$$

This result accentuates the topological character of this quantity and justifies its name.

3.1.2 Geometrical aspects of the BPS equations

One interesting aspect of this constructions is the geometrical interpretation we can give to the BPS equations. We can see a static field configuration as a path $\phi(x)$ in field space parametrized by the coordinate x. Then we have that $v_a \equiv \frac{d\phi_a}{dx}$ is a vector – the velocity, if you will – tangent to the path at each point. The BPS equations (3.15) can be rewritten as

$$v = \pm\nabla_\eta W, \tag{3.21}$$

where we call $(\nabla_\eta W)_a \equiv \eta^{-1}_{ab}\frac{\partial W}{\partial \phi_b}$ the η-gradient of W. Equation (3.21) tell us that the vector tangent to the path of a BPS state is equal, at each point, to the η-gradient of the prepotential W. That is, each solution to the BPS equations is given by a path following the η-gradient lines of W.

We have that the paths of BPS states never intersect each other, since this would mean that the η-gradient of W is multivalued. The η-gradient lines can at most meet tangentially or converge to points where $\nabla_\eta W = 0$, *i.e.*, the vacua of the potential U are sources or sinks of η-gradient lines. Since our finite energy BPS states start and finish at vacua points, this means that the paths they describe in field space connect a source to a sink of η-gradient lines. This fact implies that the prepotential W varies monotonically across the path of a given configuration. We will observe both of these behaviors explicitly in a later section when we construct BPS states for the FKZ models.

3.1.3 Lagrangian formalism

Furthermore, in order to study the dynamical aspects of these theories, we would like to have a Lagrangian representation of them. A Lagrangian that defines a theory with static energy functional of the form (3.16) is

$$\mathcal{L} = \frac{1}{2}\eta_{ab}\partial_\mu\phi_a\partial^\mu\phi_b - U(\phi) \tag{3.22}$$

giving the E-L equations

$$\eta_{ab}\left[\partial_\mu\partial^\mu\phi_b + \Gamma_{bcd}\partial_\mu\phi_c\partial^\mu\phi_d\right] + \frac{\partial U}{\partial\phi_a} = 0 \tag{3.23}$$

where

$$\Gamma_{bcd} \equiv \frac{1}{2}\eta^{-1}_{be}\left[\frac{\partial\eta_{ec}}{\partial\phi_d} + \frac{\partial\eta_{ed}}{\partial\phi_c} - \frac{\partial\eta_{cd}}{\partial\phi_e}\right]. \tag{3.24}$$

The formalism we just introduced is quite general and, in particular, the loose requirements over the matrix η allow us to try and fit in it a whole range of existing models. Moreover, it is also possible to construct theories from the ground up, by definition of a proper prepotential, and these theories come automatically equipped with the respective BPS equations and Bogomolny bounds. In the next section we use this second approach and take a look at a class of models proposed in [5], which we will call **FKZ models**, where a group theoretical approach is used in the construction of the theories.

3.2 The FKZ models

Consider a Lie algebra [10] $\mathfrak{g}$ of rank r and its simple roots α_a, $a = 1, \ldots, r$. The field $\varphi = (\phi_1, \ldots, \phi_r)$ will take values in root space:

$$\varphi \equiv \sum_{a=1}^{r}\phi_a\frac{2\,\alpha_a}{||\alpha_a||^2}. \tag{3.25}$$

In order to define the prepotential W, we need to choose a representation $\mathcal{R}$ of $\mathfrak{g}$ with weights μ_k. And so, we take the prepotential to be of the form

$$W \equiv \sum_{\mu_k \in \mathcal{R}} C_{\mu_k} \, e^{i\mu_k \cdot \varphi}. \tag{3.26}$$

We would like the prepotential to be real, that is,

$$\begin{aligned} W &= W^* \\ \Rightarrow \sum_{\mu_k \in \mathcal{R}} C_{\mu_k} \, e^{i\mu_k \cdot \varphi} &= \sum_{\mu_k \in \mathcal{R}} C^*_{\mu_k} \, e^{-i\mu_k \cdot \varphi}. \end{aligned} \tag{3.27}$$

If we stick with representations satisfying

$$\mu_k \in \mathcal{R} \Leftrightarrow -\mu_k \in \mathcal{R} \tag{3.28}$$

we can write

$$W = \sum_{\mu_k \in \mathcal{R}^{(+)}} \left(C_{\mu_k} \, e^{i\mu_k \cdot \varphi} + C_{-\mu_k} \, e^{i(-\mu_k) \cdot \varphi} \right) \tag{3.29}$$

where $\mathcal{R}^{(+)}$ stands for the fact that we only consider one weight μ_k out of each pair $(\mu_k, -\mu_k) \in \mathcal{R}$ [2]. And so the reality condition becomes

$$\begin{aligned} \sum_{\mu_k \in \mathcal{R}^{(+)}} \left(C_{\mu_k} \, e^{i\mu_k \cdot \varphi} + C_{-\mu_k} \, e^{-i\mu_k \cdot \varphi} \right) &= \sum_{\mu_k \in \mathcal{R}^{(+)}} \left(C^*_{\mu_k} \, e^{-i\mu_k \cdot \varphi} + C^*_{-\mu_k} \, e^{i\mu_k \cdot \varphi} \right) \\ \Rightarrow \sum_{\mu_k \in \mathcal{R}^{(+)}} \left[\left(C_{\mu_k} - C^*_{-\mu_k} \right) \, e^{i\mu_k \cdot \varphi} + \left(C_{-\mu_k} - C^*_{\mu_k} \right) \, e^{-i\mu_k \cdot \varphi} \right] &= 0, \end{aligned} \tag{3.30}$$

which implies that the coefficients have to satisfy

$$C_{\mu_k} = C^*_{-\mu_k}. \tag{3.31}$$

If we define $C_{\mu_k} \equiv \frac{1}{2} \left(\gamma_{\mu_k} - i\delta_{\mu_k} \right)$, such that $\gamma, \delta \in \mathbb{R}$ and

$$\gamma_{\mu_k} = \gamma_{-\mu_k} \; , \; \delta_{\mu_k} = -\delta_{-\mu_k} \tag{3.32}$$

[2]Notice that if the representation has weights with value zero that would only add a constant in the prepotential and since we are only interested in derivatives – or differences – of the prepotential, we can always ignore additive constants.

then we can write

$$
\begin{aligned}
W &= \sum_{\mu_k \in \mathcal{R}^{(+)}} \left[\frac{1}{2} \left(\gamma_{\mu_k} - i\delta_{\mu_k} \right) e^{i\mu_k \cdot \varphi} + \frac{1}{2} \left(\gamma_{-\mu_k} - i\delta_{-\mu_k} \right) e^{-i\mu_k \cdot \varphi} \right] \\
&= \sum_{\mu_k \in \mathcal{R}^{(+)}} \left[\gamma_{\mu_k} \left(\frac{e^{i\mu_k \cdot \varphi} + e^{-i\mu_k \cdot \varphi}}{2} \right) + \delta_{\mu_k} \left(\frac{e^{i\mu_k \cdot \varphi} - e^{-i\mu_k \cdot \varphi}}{2i} \right) \right] \\
\therefore\ W &= \sum_{\mu_k \in \mathcal{R}^{(+)}} \left[\gamma_{\mu_k} \cos(\mu_k \cdot \varphi) + \delta_{\mu_k} \sin(\mu_k \cdot \varphi) \right].
\end{aligned} \tag{3.33}
$$

Equation (3.33) gives the general form of the prepotential for the FKZ models, as presented in [5]. These theories can become fairly complicated very fast, so we allow ourselves a bit of simplification and consider only models for which $\delta_{\mu_k} = 0$, *i.e.*, we only consider prepotentials of the form

$$
W = \sum_{\mu_k \in \mathcal{R}^{(+)}} \gamma_{\mu_k} \cos(\mu_k \cdot \varphi). \tag{3.34}
$$

Even with this restriction the models that emerge are very rich, as we will see. The potential (3.17) for these theories will have the general form

$$
U(\phi) = \frac{1}{2} \eta^{-1}_{ab} \frac{\partial W}{\partial \phi_a} \frac{\partial W}{\partial \phi_b}, \tag{3.35}
$$

where

$$
\frac{\partial W}{\partial \phi_a} = -2 \frac{\mu_k \cdot \alpha_a}{||\alpha_a||^2} \sum_{\mu_k \in \mathcal{R}^{(+)}} \gamma_{\mu_k} \sin\left(2 \sum_b \phi_b \frac{\mu_k \cdot \alpha_b}{||\alpha_b||^2} \right). \tag{3.36}
$$

Further, the vacua will satisfy equation (3.18), that in this case reads

$$
\frac{\mu_k \cdot \alpha_a}{||\alpha_a||^2} \sum_{\mu_k \in \mathcal{R}^{(+)}} \gamma_{\mu_k} \sin\left(2 \sum_b \phi_b \frac{\mu_k \cdot \alpha_b}{||\alpha_b||^2} \right) = 0. \tag{3.37}
$$

The vacuum manifold structure will be, in general, very complex and depend heavily on the values of the constants γ_k. In particular, the points

$$
\phi_a = n_a \pi, \tag{3.38}
$$

for $n_a \in \mathbb{Z}$, will always be in the vacuum set since we have from Lie algebra theory that the weights μ_k always satisfy $\frac{2\mu_k \cdot \alpha_a}{||\alpha_a||^2} = m_{ka}$, where $m_{ka} \in \mathbb{Z}$, so that

$$
\sin\left(2\pi \sum_a m_{ka} n_a \right) = 0, \tag{3.39}
$$

as the sum of integers is an integer. Other types of vacua, that rely on further properties of

Lie algebra theory are discussed in [5]. Here we will use a brute force approach to determine the other vacua points and solve equations (3.37) for each model independently.

In [5] the authors present models based on the algebras $\mathfrak{su}(2)$, $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$. Here we present again the case $\mathfrak{su}(2)$, a theory with one field, for which we will in chapter 4 solve the time-dependent equation numerically and simulate the scattering of kinks. We also discuss the cases for the algebraas $\mathfrak{g}_2$ and $\mathfrak{su}(4)$, theories with two and three fields, respectively, and some of their static solutions. These last two cases were not explored in [5].

3.2.1 The FKZ models for the algebra $\mathfrak{su}(2)$

The algebra $\mathfrak{su}(2)$ has rank $r = 1$, thus it has only one simple root that we can normalize to be $\alpha = 1$. As root space is one-dimensional in this case, we have a single field

$$\varphi = 2\phi. \tag{3.40}$$

Besides that, the matrix η is just a number and we can make it equal to unit without loss of generality. In order to construct the prepotential we need a representation.

3.2.2 The doublet representation of $\mathfrak{su}(2)$

Let's begin with the simplest non-trivial representation, namely the doublet ($j = \frac{1}{2}$) representation. The weights are

$$\mu_1 = \frac{1}{2}\,, \;\; \mu_2 = -\frac{1}{2} = -\mu_1 \tag{3.41}$$

and the representation fulfills the requirement (3.28) for reality [3]. The prepotential (3.34) for this representation is given by

$$W = \gamma_1 \cos(\mu_1 \cdot \varphi) = \gamma_1 \cos\phi. \tag{3.42}$$

In particular, for $\gamma_1 = 1$, we have

$$\frac{dW}{d\phi} = -\sin\phi \tag{3.43}$$

which gives the BPS equations (3.15)

$$\frac{d\phi}{dx} = \pm\sin\phi. \tag{3.44}$$

This equation is equivalent to the one we got for the sine-Gordon model (2.61), only rescaled $\phi_{SG} = 2\phi_{\mathfrak{su}(2)}$. This shows that the sine-Gordon model is a particular case of this general class of theories.

[3] In fact, every irreducible representation of $\mathfrak{su}(2)$ will fulfill it, as the weights of representation j are $\mu = -j, -j+1, \ldots, j-1, j$.

3.2.3 The triplet representation of $\mathfrak{su}(2)$

For the triplet ($j = 1$) representation we have the weights

$$\mu_1 = 1\,,\ \mu_2 = 0\,,\ \mu_3 = -1 = -\mu_1 \tag{3.45}$$

which also satisfy (3.28). The prepotential then is

$$W = \gamma_1 \cos(2\phi) + \gamma_2 \Rightarrow \frac{dW}{d\phi} = -2\gamma_1 \sin(2\phi) \tag{3.46}$$

That too gives a BPS equation which is equivalent to sine-Gordon [4].

3.2.4 The irreducible representations of $\mathfrak{su}(2)$

Up until now we only got sine-Gordon-type theories with this approach. Nevertheless, things start to look more interesting for larger irreducible representations, *i.e.*, $j > 1$. For the $(2j + 1)$ dimensional representation, we have the weights

$$\mu_k = j + 1 - k\,;\ k \in \{1, \dots, 2j + 1\}. \tag{3.47}$$

For the cases where j is a half-integer we have no zero weights and the prepotential is

$$\begin{aligned} W &= \sum_{k=1}^{j+\frac{1}{2}} \gamma_k \cos(2\mu_k\phi) \\ &= \sum_{k=1}^{j+\frac{1}{2}} \gamma_k \cos(2(j + 1 - k)\phi). \end{aligned} \tag{3.48}$$

For the cases where j is an integer we have a zero weight when $k = j + 1$ in (3.47) and the prepotential will be

$$\begin{aligned} W &= \gamma_{j+1} + \sum_{k=1}^{j} \gamma_k \cos(2\mu_k\phi) \\ &= \gamma_{j+1} + \sum_{k=1}^{j} \gamma_k \cos(2(j + 1 - k)\phi). \end{aligned} \tag{3.49}$$

These give the general form of the prepotential for the irreducible representations of $\mathfrak{su}(2)$.

3.2.5 The $2 \oplus 3$ representation of $\mathfrak{su}(2)$

We made no restrictions concerning the irreducibility of the chosen representation, so a reducible representation should work just as fine as a irreducible one. Consider, *e.g.*, the

[4]Up to constants that can be absorbed through a rescale of spacetime coordinates x^μ or of the field ϕ

representation $2 \oplus 3$, the direct sum of the doublet and triplet representations. The weights in this representation are

$$\mu_1 = 1\,,\ \mu_2 = \frac{1}{2}\,,\ \mu_3 = 0\,,\ \mu_4 = -\frac{1}{2} = -\mu_2\,,\ \mu_5 = -1 = -\mu_1 \tag{3.50}$$

and the prepotential,

$$W = \gamma_1 \cos\phi + \gamma_2 \cos(2\phi). \tag{3.51}$$

Taking the derivative of W we get

$$\begin{aligned}\frac{dW}{d\phi} &= -\gamma_1 \sin\phi - 2\gamma_2 \sin(2\phi) \\ &= -\gamma_1 \sin\phi - 4\gamma_2 \sin\phi\cos\phi \\ &= -\gamma_1 \sin\phi \left(1 + b\cos\phi\right)\end{aligned} \tag{3.52}$$

where $b \equiv \frac{4\gamma_2}{\gamma_1}$. So the BPS equations are

$$\frac{d\phi}{dx} = \pm\gamma_1 \sin\phi \left(1 + b\cos\phi\right). \tag{3.53}$$

We can rescale the coordinates $x^\mu \mapsto \frac{x^\mu}{\gamma_1}$ and find

$$\frac{d\phi}{dx} = \pm \sin\phi \left(1 + b\cos\phi\right) \tag{3.54}$$

Using

$$\frac{dW}{d\phi} = \sin\phi \left(1 + b\cos\phi\right) \tag{3.55}$$

we find the potential (3.17) for this theory

$$U = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2 = \frac{1}{2}\sin^2\phi \left(1 + b\cos\phi\right)^2. \tag{3.56}$$

The vacuum set consists of the points in field space for which

$$\frac{dW}{d\phi} = \sin\phi \left(1 + b\cos\phi\right) = 0 \tag{3.57}$$

that is, each point ϕ such that

$$(i)\, \sin\phi = 0 \ \text{ or } \ (ii)\, \cos\phi = -\frac{1}{b}. \tag{3.58}$$

If $|b| < 1$, equation (ii) has no solution and the only vacua are the solutions of (i), $\phi = \pi n$,

$n \in \mathbb{Z}$. For $|b| \geq 1$, besides these, we also have the solutions of (*ii*), $\phi = 2\pi m \pm \arccos(-1/b)$, $m \in \mathbb{Z}$ [5]. Summarizing, the vacuum set $\mathcal{V}$ for this theory will be

$$\mathcal{V} = \{\pi n; n \in \mathbb{Z}\}, \text{ if } |b| \leq 1 \tag{3.59}$$

$$\mathcal{V} = \{\pi n; n \in \mathbb{Z}\} \cup \{2\pi m \pm \arccos\left(-\frac{1}{b}\right); m \in \mathbb{Z}\}, \text{ if } |b| > 1 \tag{3.60}$$

It can be seen in figure 3.1 that the region around vacuum $\phi = \pi$ flattens as $|b|$ approaches

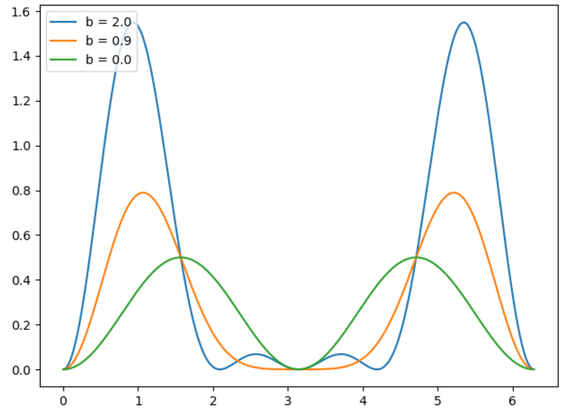


Figure 3.1: Potential $U(\phi)$ of the $\mathfrak{su}(2)$ models for $b = 0$, $b = 0.9$ and $b = 2$

1. When $|b| = 1$, we have two degenerate vacua points in $\phi = \pi$ and these points separate for $|b| > 1$, which leads to the appearence of a third vacuum between $\phi = 0$ and $\phi = \pi$. Some numerically calculated BPS states are show in figure 3.2. The numerical procedure to obtain such solutions is presented in chapter 4. Notice that for $|b| = 2$ the solution reaches vacua that do not exist for $|b| < 1$.

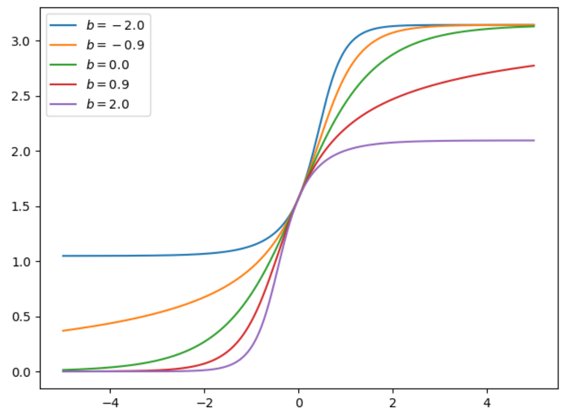


Figure 3.2: Solution of the $\mathfrak{su}(2)$ model for $|b| = 0$, $|b| = 0.9$ and $|b| = 2$

[5] Note that solutions of both (*i*) and (*ii*) coincide when $|b| = 1$.

3.2.6 The FKZ models for the algebra $\mathfrak{g}_2$

The algebra $\mathfrak{g}_2$ is of rank $r = 2$, thus we will have two fields ϕ_1 and ϕ_2

$$\varphi = \phi_1 \frac{2\alpha_1}{||\alpha_1||^2} + \phi_2 \frac{2\alpha_2}{||\alpha_2||^2}. \tag{3.61}$$

The Cartan matrix of this algebra is given by

$$K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{3.62}$$

We choose a normalization such that $||\alpha_1||^2 = 1$ and $||\alpha_2||^2 = 3$, where we have used the fact that the norms of the simple roots satisfy

$$\frac{||\alpha_a||^2}{||\alpha_b||^2} = \frac{K_{ab}}{K_{ba}}, \tag{3.63}$$

We define η to be

$$\eta = \begin{pmatrix} 4 & -2\lambda \\ -2\lambda & 4/3 \end{pmatrix} \tag{3.64}$$

Thus its inverse will be

$$\eta^{-1} = \frac{1}{8 - 6\lambda^2} \begin{pmatrix} 2 & 3\lambda \\ 3\lambda & 6 \end{pmatrix} \tag{3.65}$$

where we have introduced a parameter λ in a such way that $\eta_{ab}|_{\lambda=1} = 2K_{ab}/\, ||\alpha_a||^2$. In order for η to be positive definite – and consequently also invertible –, its eigenvalues have to satisfy

$$\omega = 4 \pm \sqrt{4 + 9\lambda^2} \geq 0, \tag{3.66}$$

which lead us to restrict the parameter λ to values

$$|\lambda| < \frac{2}{\sqrt{3}}. \tag{3.67}$$

The last step in the construction of the model is to choose a representation for this algebra. A good starting point is to consider the fundamental representations, *i.e.*, representations for which the highest weight is a fundamental one. For an algebra of rank r, the fundamental weights are given by

$$\lambda_a = \sum_{b=1}^{r} K^{-1}_{ab} \alpha_b \tag{3.68}$$

where K^{-1} is the inverse of the Cartan matrix. So, in the case of $\mathfrak{g}_2$, we have the two fundamental weights

$$\lambda_1 = 2\alpha_1 + \alpha_2 \quad \text{and} \quad \lambda_2 = 3\alpha_1 + 2\alpha_2. \tag{3.69}$$

The first fundamental representation, *i.e.*, the representation with highest weight λ_1, has the following weights

$$\begin{aligned}
\mu_1 &= \lambda_1 = 2\alpha_1 + \alpha_2 \\
\mu_2 &= \lambda_1 - \alpha_1 = \alpha_1 + \alpha_2 \\
\mu_3 &= \lambda_1 - \alpha_1 - \alpha_2 = \alpha_1 \\
\mu_4 &= \lambda_1 - 2\alpha_1 - \alpha_2 = 0 \\
\mu_5 &= \lambda_1 - 3\alpha_1 - \alpha_2 = -\alpha_1 = -\mu_3 \\
\mu_6 &= \lambda_1 - 3\alpha_1 - 2\alpha_2 = -\alpha_1 - \alpha_2 = -\mu_2 \\
\mu_7 &= \lambda_1 - 4\alpha_1 - 2\alpha_2 = -2\alpha_1 - \alpha_2 = -\mu_1
\end{aligned} \tag{3.70}$$

and already satisfies the requirement (3.28) for the reality of the prepotential W.

In order to calculate the internal products $\mu_k \cdot \varphi$, which are passed as arguments for the cosines in the prepotential, it is important to remember some relations between simple roots, fundamental weights and the Cartan matrix. The Cartan matrix is constructed from the simple roots as

$$\frac{2\alpha_a \cdot \alpha_b}{||\alpha_b||^2} = K_{ab}. \tag{3.71}$$

The simple roots and the fundamental weights satisfy the orthogonality condition

$$\frac{2\lambda_a \cdot \alpha_b}{||\alpha_b||^2} = \delta_{ab}. \tag{3.72}$$

Therefore, we have

$$\begin{aligned}
\mu_1 \cdot \varphi &= \sum_a \phi_a \frac{2\lambda_1 \cdot \alpha_a}{||\alpha_a||^2} = \sum_a \phi_a \delta_{1a} = \phi_1 \\
\mu_2 \cdot \varphi &= \sum_a \phi_a \frac{2(\lambda_1 - \alpha_1) \cdot \alpha_a}{||\alpha_a||^2} = \phi_1 - K_{11}\phi_1 - K_{12}\phi_2 = -\phi_1 + \phi_2 \\
\mu_3 \cdot \varphi &= \sum_a \phi_a \frac{2(\lambda_1 - \alpha_1 - \alpha_2) \cdot \alpha_a}{||\alpha_a||^2} = \phi_1 - K_{11}\phi_1 - K_{12}\phi_2 - K_{21}\phi_1 - K_{12}\phi_2 = 2\phi_1 - \phi_2
\end{aligned} \tag{3.73}$$

which gives the prepotential

$$W = \gamma_1 \cos\phi_1 + \gamma_2 \cos(\phi_1 - \phi_2) + \gamma_3 \cos(2\phi_1 - \phi_2) \tag{3.74}$$

The components of its gradient in field space will be

$$
\begin{aligned}
\frac{\partial W}{\partial \phi_1} &= -\gamma_1 \sin\phi_1 - \gamma_2 \sin(\phi_1 - \phi_2) - 2\gamma_3 \sin(2\phi_1 - \phi_2) \\
\frac{\partial W}{\partial \phi_2} &= \gamma_2 \sin(\phi_1 - \phi_2) + \gamma_3 \sin(2\phi_1 - \phi_2).
\end{aligned} \tag{3.75}
$$

Thus, the potential (3.17) for these theories will be

$$
U(\phi) = \frac{1}{8 - 6\lambda^2}\left[\left(\frac{\partial W}{\partial \phi_1}\right)^2 + 3\lambda \frac{\partial W}{\partial \phi_1}\frac{\partial W}{\partial \phi_2} + 3\left(\frac{\partial W}{\partial \phi_2}\right)^2\right] \tag{3.76}
$$

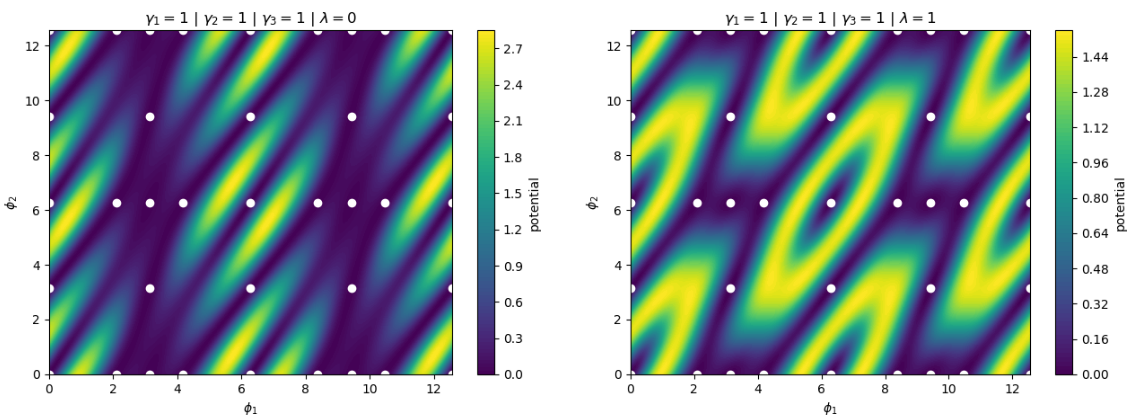
The vacua are given by the critical points of the prepotential, as in equation (3.18), that is, the points $\phi_0 = (\phi_1, \phi_2)$ which satisfy

$$
\begin{aligned}
\gamma_1 \sin\phi_1 + \gamma_2 \sin(\phi_1 - \phi_2) + 2\gamma_3 \sin(2\phi_1 - \phi_2) &= 0 \\
\gamma_2 \sin(\phi_1 - \phi_2) + \gamma_3 \sin(2\phi_1 - \phi_2) &= 0.
\end{aligned} \tag{3.77}
$$

The set of vacua points will depend on the choice of the parameters γ_i. For the particular choice $\gamma_i = 1$, for example, the vacua are

$$
\phi_0 = \begin{cases} (n_1\pi, n_2\pi) \\ (\frac{2\pi}{3} + 2\pi n_1, 2\pi n_2) \\ (\frac{4\pi}{3} + 2\pi n_1, 2\pi n_2) \end{cases} \tag{3.78}
$$

where $n_1, n_2 \in \mathbb{Z}$. Note that the vacuum set does not depend on the values of the parameter λ, even though it appears explicitly in the expression for the potential (3.76). Plots for this potential, with the choice $\gamma_i = 1$, are given in figure 3.3.



(a) The potential with $\lambda = 0$.

(b) The potential with $\lambda = 1$

Figure 3.3: Plot of the potential $U(\phi)$ with parameters $\gamma_i = 1$ and (a) $\lambda = 0$ and (b) $\lambda = 1$. White dots give the vacua points. Their position is independent of λ

Thus, the BPS equations read

$$\begin{aligned}
\frac{d\phi_1}{dx} &= \pm\left(\eta_{11}^{-1}\frac{\partial W}{\partial\phi_1} + \eta_{12}^{-1}\frac{\partial W}{\partial\phi_2}\right)\\
&= \pm\frac{1}{8-6\lambda^2}\left(2\frac{\partial W}{\partial\phi_1} + 3\lambda\frac{\partial W}{\partial\phi_2}\right)\\
\therefore\ \frac{d\phi_1}{dx} &= \pm\frac{1}{8-6\lambda^2}\Big[-2\gamma_1\sin\phi_1 + (3\lambda-2)\gamma_2\sin(\phi_1-\phi_2)+\\
&\qquad +(3\lambda-4)\gamma_3\sin(2\phi_1-\phi_2)\Big]
\end{aligned} \tag{3.79}$$

and

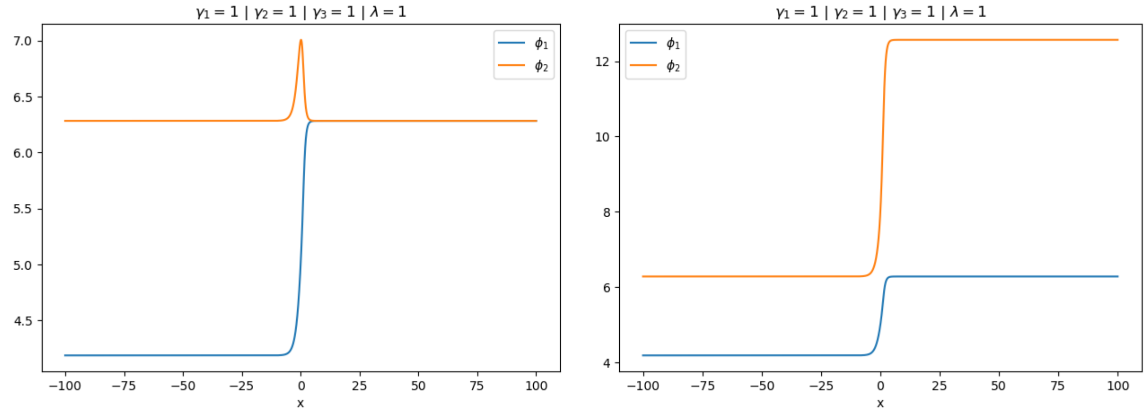
$$\begin{aligned}
\frac{d\phi_2}{dx} &= \pm\left(\eta_{21}^{-1}\frac{\partial W}{\partial\phi_1} + \eta_{22}^{-1}\frac{\partial W}{\partial\phi_2}\right)\\
&= \pm\frac{1}{8-6\lambda^2}\left(3\lambda\frac{\partial W}{\partial\phi_1} + 6\frac{\partial W}{\partial\phi_2}\right)\\
\therefore\ \frac{d\phi_2}{dx} &= \pm\frac{1}{8-6\lambda^2}\Big[-3\lambda\gamma_1\sin\phi_1 + (6-3\lambda)\gamma_2\sin(\phi_1-\phi_2)+\\
&\qquad +(6-6\lambda)\gamma_3\sin(2\phi_1-\phi_2)\Big].
\end{aligned} \tag{3.80}$$

These equations must be solved numerically and some results are presented in the next section. The method used in these calculations will be explored in chapter 4. Summarizing, it consists in choosing an initial point in field space $\phi(0) = (\phi_1(0), \phi_2(0))$ and let it evolve from there in both directions, $x \to \pm\infty$, following the η-gradient lines of the prepotential W. As long as we do not choose a vacuum point, the flow of the η-gradient is uniquely defined and we should get a non-trivial configuration connecting two vacua.

3.2.7 Numerical support

All the simulations presented here were obtained for the particular choices $\gamma_i = 1$ and $\lambda = 1$. The potential $U(\phi)$ for this model is shown in figure 3.3b. Different values for these parameters were also considered in other simulations, however, we observed that the general properties of the solutions did not differ too much from model to model.

In figure 3.4 we have particular configurations obtained for close initial conditions $\phi(0)$. The configuration in figure 3.4a was initialized at the point $\phi(0) = (5, 7)$. Here we can already oberve a severe difference from the models with a single field: while for ϕ_1 we have a usual kink profile, for ϕ_2 we have a single bump. If analysing the two profiles independently, one could have the false impression that ϕ_2 is topologically trivial. Nevertheless it is important to reinforce the idea that the two fields must not be taken separately, as they are just components of the fundamental field φ. Moreover, note that the configuration as a whole interpolates between vacua $(4\pi/3, 2\pi)$ and $(2\pi, 2\pi)$, *i.e.*, the configuration has indeed non-trivial topological data. In figure 3.4b the configuration was initialized at $\phi(0) = (5, 8)$ and as a result we obtained kink profiles for both ϕ_1 and ϕ_2. This time the solution interpolates between vacua $(4\pi/3, 2\pi)$ and $(2\pi, 4\pi)$.

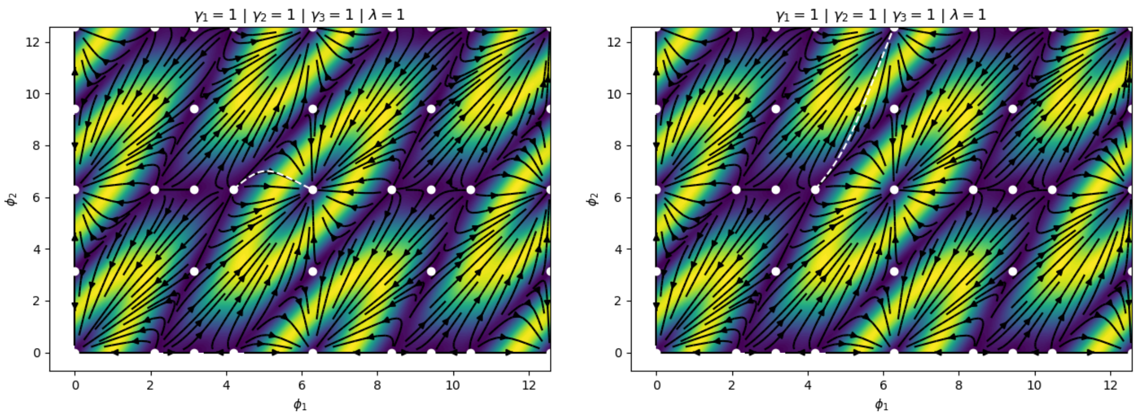


(a) The fields ϕ_1 and ϕ_2 have different values at $x = -\infty$ and go to the same vacuum at $x = +\infty$.

(b) Both fields behave essentially like kinks, interpolating different vacua.

Figure 3.4: Two numerically calculated BPS states of the model with $\lambda = 1$ and $\gamma_i = 1$.

In figure 3.5 we have the η-gradient lines of the prepotential W plotted over the potential U, in colors. The white dashed lines indicate the paths described the solutions in figure 3.4. Notice that the paths follow the η-gradient flow, as it was discussed in section 3.1.2. Here we can understand why the configurations in 3.4 go to different vacua. At the point $\phi = (5, 7)$ the η-gradient flow connects the vacua $(4\pi/3, 2\pi)$ and $(4\pi/3, 2\pi)$, which lie on the same horizontal line $\phi_2 = 2\pi$. So, from this point of view, it is expected for the profile of ϕ_2 to have a bump, since it will have to return to the same value of the ϕ_2 coordinate at $x = \infty$.



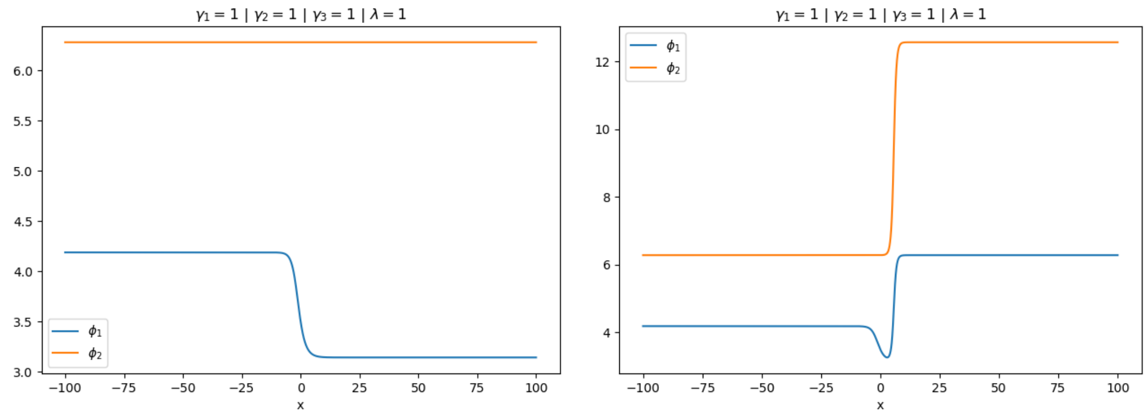
(a) A solution interpolating the vacua $(\frac{4\pi}{3}, 2\pi)$ and $(2\pi, 2\pi)$.

(b) A solution interpolating the vacua $(\frac{4\pi}{3}, 2\pi)$ and $(2\pi, 4\pi)$.

Figure 3.5: The $\nabla_\eta W$ lines are plotted as black arrows over the potential U, in colors. The paths described by the solutions in figure 3.4 are plotted as white dashed lines.

An even more dramatic change in profiles was obtained for initial condition $\phi(0) =$

$(7\pi/6, 2\pi)$, in figure 3.6a, and $\phi(0) = (7\pi/6, 2\pi + 10^{-3})$, in figure 3.6b. In this case, even a slight change of order 10^{-3} in the initial conditions was enough to completely change the configurations and the vacua they interpolate. In figure 3.6a while ϕ_1 has a kink profile, ϕ_2 remain constant and the configurations interpolates points $(4\pi/3, 2\pi)$ and $(\pi, 2\pi)$. In figure 3.6b we have basically two kink profiles interpolating $(4\pi/3, 2\pi)$ and $(2\pi, 4\pi)$, with the exception that ϕ_1 presents a little bump before tunneling to the other vacuum.



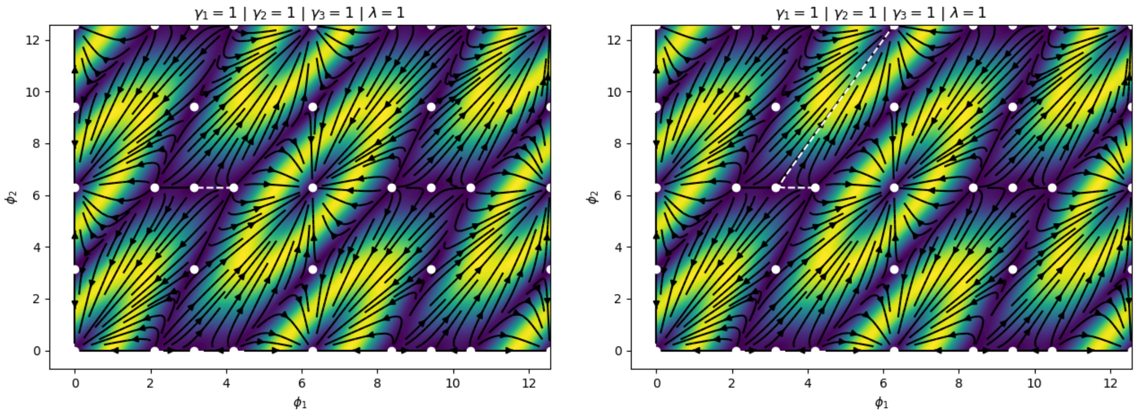
(a) A solution interpolating the vacua $(\frac{4\pi}{3}, 2\pi)$ and $(\pi, 2\pi)$.

(b) A solution interpolating the vacua $(\frac{4\pi}{3}, 2\pi)$ and $(2\pi, 4\pi)$.

Figure 3.6: BPS states constructed using slightly different initial conditions. A small change of order 10^{-3} in the value of ϕ_2 implies a dramatic change in the resulting configurations.

In figures 3.7 we have plotted the paths described by the configurations in 3.6. Notice that between vacua $(\frac{4\pi}{3}, 2\pi)$ and $(\pi, 2\pi)$ there is exactly one η-gradient line connecting the two. Any variation, no matter how small, in vertical axis ϕ_2 would take us to a region where the η-gradient flows opposite to vacuum $(\pi, 2\pi)$. This exactly what was observed in the case for figures 3.6b and 3.7b.

Moreover, notice that both solutions 3.5b and 3.7b are BPS states which interpolates vacua $(4\phi/3, 2\pi)$ and $(2\pi, 4\pi)$. This means both have the same topological charge (3.20) and energy, given by the Bogomolny bound (3.12), namely, $E = |Q| = 9/2$. There are in fact an infinity of paths connecting these two vacua following the η-gradient flow, all of them described by BPS states with exactly the same energy and topological data. In principle, then, we do not have any topological arguments forbidding transition from one of these configurations to another, or energetic arguments demonstrating a preference for a specific one. We were not yet able to study perturbations around these solutions, though it is in our plans as a continuation for this work.



(a) A configuration interpolating the vacua $(\frac{4\pi}{3}, 2\pi)$ and $(\pi, 2\pi)$.

(b) A slight change in the initial condition for ϕ_2 changes the configuration completely, which now interpolates $(\frac{4\pi}{3}, 2\pi)$ and $(2\pi, 4\pi)$

Figure 3.7: The paths decribed by the solution given in figure 3.6.

3.2.8 The FKZ models for the algebra $\mathfrak{su}(4)$

The algebra $\mathfrak{su}(4)$ is of rank $r = 3$, thus the field space will be three-dimensional and the field is a scalar triplet:

$$\varphi = \phi_1 \frac{2\alpha_1}{||\alpha_1||^2} + \phi_2 \frac{2\alpha_2}{||\alpha_2||^2} + \phi_3 \frac{2\alpha_3}{||\alpha_3||^2}. \tag{3.81}$$

The Cartan matrix of the algebra $\mathfrak{su}(4)$ is

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{3.82}$$

In constructing the FKZ potential for this algebra we choose the matrix η to be

$$\eta = \begin{pmatrix} 2 & -\lambda & 0 \\ -\lambda & 2 & -\lambda \\ 0 & -\lambda & 2 \end{pmatrix}, \tag{3.83}$$

such that $\eta_{ab}|_{\lambda=1} = K_{ab}/\,||\alpha_a||^2$. Its inverse reads

$$\eta^{-1} = \frac{1}{8 - 4\lambda^2} \begin{pmatrix} 4-\lambda^2 & 2\lambda & \lambda^2 \\ 2\lambda & 4 & 2\lambda \\ \lambda^2 & 2\lambda & 4-\lambda^2 \end{pmatrix}. \tag{3.84}$$

The parameter λ should be kept in the interval

$$|\lambda| < \sqrt{2} \tag{3.85}$$

to ensure that η is positive definite and invertible.

The fundamental weights of $\mathfrak{su}(4)$ are given by

$$\begin{aligned}
\lambda_1 &= \frac{1}{4}\left(3\alpha_1 + 2\alpha_2 + \alpha_3\right) \\
\lambda_2 &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \alpha_3\right) \\
\lambda_3 &= \frac{1}{4}\left(\alpha_1 + 2\alpha_2 + 3\alpha_3\right).
\end{aligned} \tag{3.86}$$

The weights of the first fundamental representation are given by

$$\begin{aligned}
\mu_1 &= \lambda_1 = \frac{1}{4}\left(3\alpha_1 + 2\alpha_2 + \alpha_3\right) \\
\mu_2 &= \lambda_1 - \alpha_1 = \frac{1}{4}\left(-\alpha_1 + 2\alpha_2 + \alpha_3\right) \\
\mu_3 &= \lambda_1 - \alpha_1 - \alpha_2 = \frac{1}{4}\left(-\alpha_1 - 2\alpha_2 + \alpha_3\right) \\
\mu_4 &= \lambda_1 - \alpha_1 - \alpha_2 - \alpha_3 = -\frac{1}{4}\left(\alpha_1 + 2\alpha_2 + 3\alpha_3\right)
\end{aligned} \tag{3.87}$$

those of the second fundamental representation are

$$\begin{aligned}
\tilde{\mu}_1 &= \lambda_2 = \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \alpha_3\right) \\
\tilde{\mu}_2 &= \lambda_2 - \alpha_2 = \frac{1}{2}\left(\alpha_1 + \alpha_3\right) \\
\tilde{\mu}_3 &= \lambda_2 - \alpha_1 - \alpha_2 = \frac{1}{2}\left(-\alpha_1 + \alpha_3\right) \\
\tilde{\mu}_4 &= \lambda_2 - \alpha_2 - \alpha_3 = \frac{1}{2}\left(\alpha_1 - \alpha_3\right) = -\tilde{\mu}_3 \\
\tilde{\mu}_5 &= \lambda_2 - \alpha_1 - \alpha_2 - \alpha_3 = -\frac{1}{2}\left(\alpha_1 + \alpha_3\right) = -\tilde{\mu}_2
\end{aligned} \tag{3.88}$$

and, finaly, the weights of the third fundamental representation are

$$\begin{aligned}
\bar{\mu}_1 &= \lambda_3 = \frac{1}{4}\left(\alpha_1 + 2\alpha_2 + 3\alpha_3\right) = -\mu_4 \\
\bar{\mu}_2 &= \lambda_3 - \alpha_3 = \frac{1}{4}\left(\alpha_1 + 2\alpha_2 - \alpha_3\right) = -\mu_3 \\
\bar{\mu}_3 &= \lambda_3 - \alpha_3 - \alpha_2 = \frac{1}{4}\left(\alpha_1 - 2\alpha_2 - \alpha_3\right) = -\mu_2 \\
\bar{\mu}_4 &= \lambda_3 - \alpha_3 - \alpha_2 - \alpha_1 = -\frac{1}{4}\left(3\alpha_1 + 2\alpha_2 + \alpha_3\right) = -\mu_1.
\end{aligned} \tag{3.89}$$

None of the fundamental representations alone satisfy the requirement for reality of the prepotential, but we notice that a direct sum of the first and third, $4 \oplus \bar{4}$, does. As we only need one of each pair of weights in order to construct the prepotential, let us use the weights of representation 4 and calculate the internal products with the field φ:

$$\begin{aligned}
\mu_1 \cdot \varphi &= \sum_{a=1}^{3} \phi_a \frac{2\lambda_1 \cdot \alpha_a}{||\alpha_a||^2} = \sum_{a=1}^{3} \phi_a \delta_{1a} = \phi_1 \\
\mu_2 \cdot \varphi &= \sum_{a=1}^{3} \phi_a \frac{2(\lambda_1 - \alpha_1) \cdot \alpha_a}{||\alpha_a||^2} = \phi_1 - K_{11}\phi_1 - K_{12}\phi_2 - K_{13}\phi_3 = \phi_2 - \phi_1 \\
\mu_3 \cdot \varphi &= \sum_{a=1}^{3} \phi_a \frac{2(\lambda_1 - \alpha_1 - \alpha_2) \cdot \alpha_a}{||\alpha_a||^2} = \phi_2 - \phi_1 - K_{21}\phi_1 - K_{22}\phi_2 - K_{23}\phi_3 = \phi_3 - \phi_2 \\
\mu_4 \cdot \varphi &= \sum_{a=1}^{3} \phi_a \frac{2(\lambda_1 - \alpha_1 - \alpha_2 - \alpha_3) \cdot \alpha_a}{||\alpha_a||^2} = \phi_3 - \phi_2 - K_{31}\phi_1 - K_{32}\phi_2 - K_{33}\phi_3 = -\phi_3.
\end{aligned} \tag{3.90}$$

Then, the prepotential for the $\mathfrak{su}(4)$ FKZ model for the representation $4 \oplus \bar{4}$ is given by

$$W = \gamma_1 \cos\phi_1 + \gamma_2 \cos(\phi_1 - \phi_2) + \gamma_3 \cos(\phi_2 - \phi_3) + \gamma_4 \cos\phi_3. \tag{3.91}$$

And the components of the gradient of the prepotential in the field space are

$$\begin{aligned}
\frac{\partial W}{\partial \phi_1} &= -\gamma_1 \sin\phi_1 - \gamma_2 \sin(\phi_1 - \phi_2) \\
\frac{\partial W}{\partial \phi_2} &= \gamma_2 \sin(\phi_1 - \phi_2) - \gamma_3 \sin(\phi_2 - \phi_3) \\
\frac{\partial W}{\partial \phi_3} &= \gamma_3 \sin(\phi_2 - \phi_3) - \gamma_4 \sin\phi_3.
\end{aligned} \tag{3.92}$$

The potential for this model is given by

$$\begin{aligned}
U(\phi) = \frac{1}{16 - 8\lambda^2} \Bigg[&(4 - \lambda^2) \left(\frac{\partial W}{\partial \phi_1}\right)^2 + 4 \left(\frac{\partial W}{\partial \phi_2}\right)^2 + (4 - \lambda^2) \left(\frac{\partial W}{\partial \phi_3}\right)^2 + \\
&+ 4\lambda \frac{\partial W}{\partial \phi_1}\frac{\partial W}{\partial \phi_2} + 4\lambda \frac{\partial W}{\partial \phi_2}\frac{\partial W}{\partial \phi_3} + 2\lambda^2 \frac{\partial W}{\partial \phi_3}\frac{\partial W}{\partial \phi_1} \Bigg]
\end{aligned} \tag{3.93}$$

The vacuum set is given by the collection of points that make (3.93) vanish. In particular, these points are the solutions for the system of equations

$$\frac{\partial W}{\partial \phi_a} = 0, \tag{3.94}$$

where the derivatives are given in (3.92). Unlike the cases for algebras $\mathfrak{su}(2)$ and $\mathfrak{g}_2$, where the vacuum set was a discrete set, here it appears to have both discrete and continuous components. As in every FZK model, the points $\phi_a = n_a\pi$, $n_a \in \mathbb{Z}$, form always a discrete component of the vacuum set. Nonetheless, for the particular choice $\gamma_i = 1$, notice that the lines $\phi = (\phi_3 - \pi, \pi + 2\pi n, \phi_3)$, where $n \in \mathbb{Z}$ and for each $\phi_3 \in \mathbb{R}$, also satisfy the set of equations (3.94). This means that the vacuum set has at least some continuous components. Moreover, because of the symmetry $\phi_1 \leftrightarrow \phi_3$ in the prepotential (3.91), we also have that the lines $\phi = (\phi_1, \pi + 2\pi n, \phi_1 - \pi)$ also are in the vacuum set.

Although a thorough analysis of the vacuum set would be fundamental to the complete description of the theory and its possible solutions, here our numerical approach is in fact independent of the previous knowledge of the complete set. This is due to the fact that the starting point for the numerical integration does not need to be a vacuum point, in fact we could use random values for the initial condition $\phi(0)$, see chapter 4. Further, as will be seen below, all of our numerical experiments gave solutions interpolating the usual $\phi_a = n_a\pi$ vacua.

The BPS equations read

$$\begin{aligned}
\frac{d\phi_1}{dx} &= \pm\left(\eta^{-1}_{11}\frac{\partial W}{\partial\phi_1} + \eta^{-1}_{12}\frac{\partial W}{\partial\phi_2} + \eta^{-1}_{13}\frac{\partial W}{\partial\phi_3}\right) \\
&= \pm\frac{1}{8-4\lambda^2}\left((4-\lambda^2)\frac{\partial W}{\partial\phi_1} + 2\lambda\frac{\partial W}{\partial\phi_2} + \lambda^2\frac{\partial W}{\partial\phi_3}\right) \\
&= \pm\frac{1}{8-4\lambda^2}\Big[(\lambda^2-4)\gamma_1\sin\phi_1 + (\lambda^2+2\lambda-4)\gamma_2\sin(\phi_1-\phi_2) + \\
&\quad + (\lambda^2-2\lambda)\gamma_3\sin(\phi_2-\phi_3) - \lambda^2\gamma_4\sin\phi_3\Big]
\end{aligned} \tag{3.95}$$

$$\begin{aligned}
\frac{d\phi_2}{dx} &= \pm\left(\eta^{-1}_{21}\frac{\partial W}{\partial\phi_1} + \eta^{-1}_{22}\frac{\partial W}{\partial\phi_2} + \eta^{-1}_{23}\frac{\partial W}{\partial\phi_3}\right) \\
&= \pm\frac{1}{8-4\lambda^2}\Big[-2\lambda\gamma_1\sin\phi_1 + (4-2\lambda)\gamma_2\sin(\phi_1-\phi_2) + \\
&\quad + (2\lambda-4)\gamma_3\sin(\phi_2-\phi_3) - 2\lambda\gamma_4\sin\phi_3\Big]
\end{aligned} \tag{3.96}$$

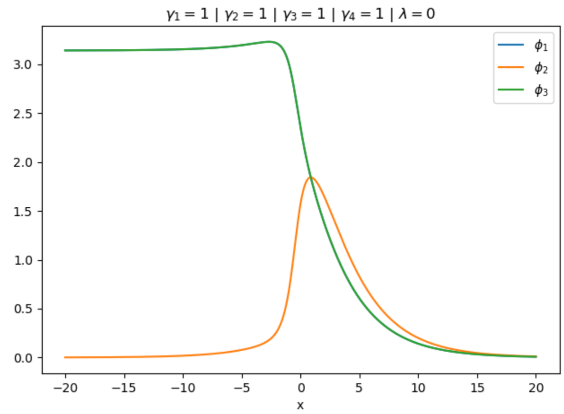
$$\begin{aligned}
\frac{d\phi_3}{dx} &= \pm\left(\eta^{-1}_{31}\frac{\partial W}{\partial\phi_1} + \eta^{-1}_{32}\frac{\partial W}{\partial\phi_2} + \eta^{-1}_{33}\frac{\partial W}{\partial\phi_3}\right) \\
&= \pm\frac{1}{8-4\lambda^2}\Big[-\lambda^2\gamma_1\sin\phi_1 + (2\lambda-\lambda^2)\gamma_2\sin(\phi_1-\phi_2) + \\
&\quad + (4-2\lambda-\lambda^2)\gamma_3\sin(\phi_2-\phi_3) + (\lambda^2-4)\gamma_4\sin\phi_3\Big].
\end{aligned} \tag{3.97}$$

This equations were solved numerically and we present some results in the next section.

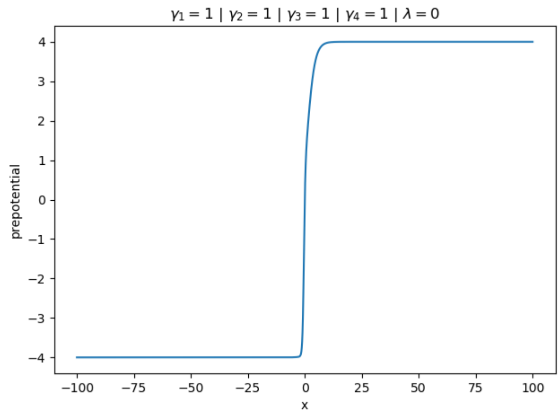
3.2.9 Numerical support

For this model we did not have the aid of plotting the η-gradient lines of the prepotential in order to check the results obtained. We, then, resorted to another result presented in section 3.1.2 claimming that the prepotential W calculated over a path ϕ described by a BPS state should be monotonic in x.

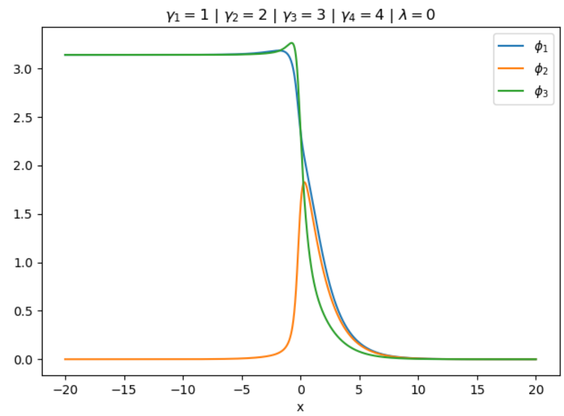
In figure 3.8a we have a configuration obtained for the choice of parameters $\gamma_i = 1$ and $\lambda = 0$, the initial condition was $\phi(0) = (3\pi/4, \pi, 3\pi/4)$. Note that we can exchange $\phi_1 \leftrightarrow \phi_3$, as it was expected by the symmetry of the prepotential W. The prepotential in 3.8b is indeed monotonic when evaluated at the path $\phi(x)$ and resembles a kink. In figure 3.8c we manage to break the symmetry of exchange $\phi_1 \leftrightarrow \phi_3$ by choosing different parameters γ_i. The configuration still interpolates the same vacua as 3.8a.



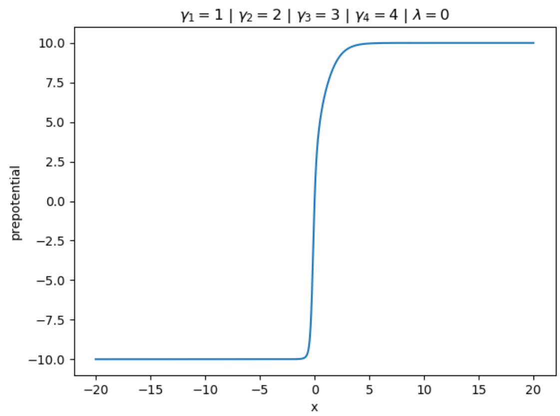
(a) A configuration interpolating vacua $(\pi, 0, \pi)$ and $(0, 0, 0)$.



(b) The prepotential for the configuration in (a).



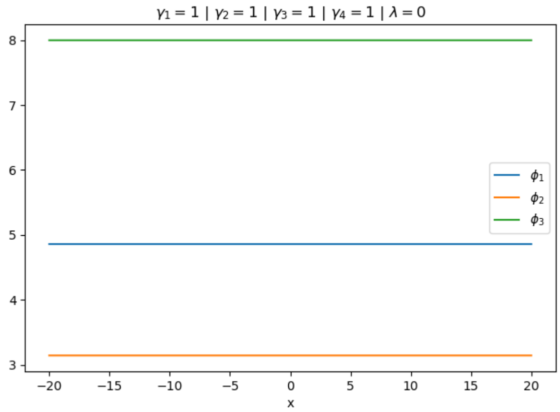
(c) A configuration interpolating vacua $(\pi, 0, \pi)$ and $(0, 0, 0)$.



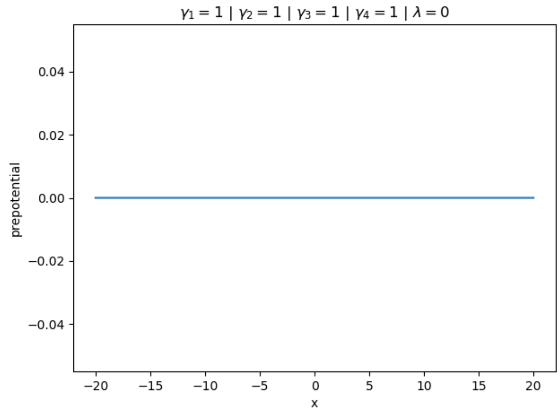
(d) The prepotential for the configuration in (c).

Figure 3.8: BPS states for the choices (a) $\gamma_i = 1$ and $\lambda = 0$, and (c) $\gamma_i = i$ and $\lambda = 0$. The two simulations were started with the same initial conditions, but the choice of γ_i in (c) broke the symmetry $\phi_1 \leftrightarrow \phi_3$.

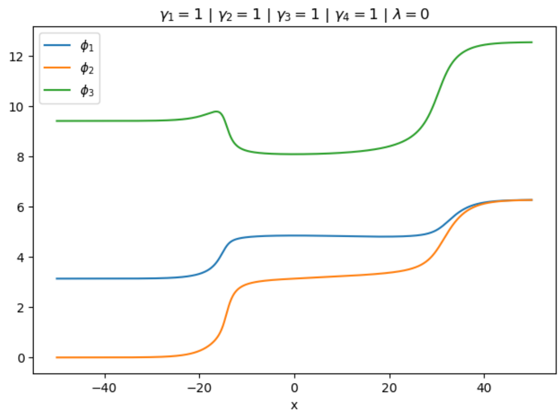
In figure 3.9a we choose as initial condition the point $\phi(0) = (8 - \pi, \pi, 8)$, which lies in the line of vacua $(\phi_3 - \pi, \pi, \phi_3)$. The configuration indeed stayed at the same point as it was expected, confirming the fact that each point in this line is a vacuum. For figure 3.9c we initiated the numerical integration at the point $\phi(0) = (8 - \pi, \pi, 8.1)$, slightly apart from the line. This was sufficient to completely change the resulting configuration. The solution now interpolates the vacua $(\pi, 0, 3\pi)$ and $(2\pi, 2\pi, 4\pi)$. The prepotential in 3.9d is indeed monotonic, but this time it resembles a 2-kink configuration.



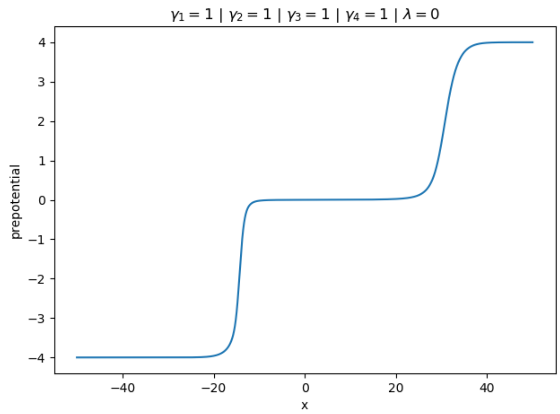
(a) A vacuum configuration.



(b) The prepotential corresponding to (a) is constant.



(c) A configuration interpolating vacua $(\pi, 0, 3\pi)$ and $(2\pi, 2\pi, 4\pi)$.



(d) The prepotential for the configuration on (c).

Figure 3.9: Numerical test around the vacua line $\phi = (\phi_3 - \pi, \pi, \phi_3)$. In (a) the initial condition was a point of the line. In (b) we started at a point slightly apart from it.

Moreover, we did not observe in any of our numerical experiments a configuration interpolating vacua different from the integer multiples of π. A possible explanation for this would be that the η-gradient of W is parallel to the lines $\phi = (\phi_3 - \pi, \pi, \phi_3)$ at their neighborhoods. This way, if no point in these lines is a source or sink of gradient lines, we indeed should not expect a non-trivial configuration to reach them. We are still working on this hypothesis, as we are in the analysis of perturbations around the static solutions for this model.

Chapter 4

Numerical methods in field theories

In this chapter we discuss numerical techniques [11] for solving first and second order differential equations. We used this techniques in the numerical simulations of the static configurations for the FKZ models in the last chapter. We will also present how to use them in the simulation of scattering processes of multi-soliton solutions of the $\mathfrak{su}(2)$ FKZ model.

4.1 Static equations

The static configurations we are interested in are solutions of the BPS equations, which are first order equations of the form [1]

$$\frac{d\phi_a}{dx} = \eta^{-1}_{ab} \frac{\partial W}{\partial \phi_b} \equiv F_a(\phi, x) \tag{4.1}$$

We discretize the parameter x into a vector of N components, $(x(i))$, using an equally spaced grid with step size h. The fields themselves become vectors

$$\phi_a(i) \equiv \phi_a(x(i)). \tag{4.2}$$

In order to compute the solutions of equations (4.1) we will use a simple Euler scheme. We discretize the derivative using a forward difference

$$\frac{dy}{dx}(i) = \frac{y(i+1) - y(i)}{h} \tag{4.3}$$

and the equations become a recurrence relations

$$\phi_a(i+1) = \phi_a(i) + hF_a(i) \tag{4.4}$$

where $F_a(i) \equiv F_a(\phi(x(i)), x(i))$. Given the initial conditions $\phi_a(0)$, we use (4.4) to calculate $\phi_a(1)$, use $\phi_a(1)$ to calculate $\phi_a(2)$, so on and so forth. So, iteratively, we can calculate the

[1]We chose the positive sign just as an example, all the results hold for the equations with negative sign as well.

fields at each point $x(i)$ – by using a `for` loop, for example.

Notice, though, that if we give as initial conditions values which are vacua of the potential, that is,

$$F_a(0) = 0 \tag{4.5}$$

then we would get $\phi_a(i) = \phi_a(i)$ and the recurrence relation would become $\phi_a(i) = \phi_a(0)$, *i.e.*, we would get a vacuum configuration as a solution. Therefore, we would like that at least some of the fields would not start at vacuum points. One thing we can do is to start the fields at values near the vacua $\phi_a(0) = \phi_a^{(vac)} + \epsilon$, this way $F_a(0)$ would be slightly different from zero, which should be enough to generate a non-trivial solution. This method, nonetheless, has at least two setbacks. The first one is that it is not guaranteed it will give you a non-trivial solution. Sometimes, the fields simply evolve back to the vacua they are close to. The second one is that you cannot predict where your configuration will be centered. One way to avoid those problems is to begin the integration at another point that is not $i = 0$. We can, *e.g.*, start the integration at $i = N/2$ and choose for initial conditions $\phi_a(N/2)$ points that are halfway between two vacua. Thereafter, we iterate the equation forward until $i = N-1$ and backward until $i = 0$, that is, we use the recurrence relations

$$\phi_a(i+1) = \phi_a(i) + hF_a(i); \;\; i = N/2, \cdots, N-1 \tag{4.6}$$

$$\phi_a(i-1) = \phi_a(i) - hF_a(i); \;\; i = N/2, \cdots, 0 \tag{4.7}$$

where we used a backward difference

$$\frac{dy}{dx}(i) = \frac{y(i) - y(i-1)}{h} \tag{4.8}$$

in equation (4.1) in order to determine the backward recurrence relation. With this approach, we manage to avoid the configurations from staying in the vacua, since they are far from them, and can choose roughly where the configuration will be centered by the choice of the initial index i.

After the computation, when we have our configuration $\phi_a(i)$, one thing we might want to do is calculate some physical quantities, such as the energy density of the solution. The static energy, which is given by

$$\varepsilon = \frac{1}{2}\eta_{ab}^{-1}\frac{d\phi_a}{dx}\frac{d\phi_b}{dx} + U(\phi) \tag{4.9}$$

can also discretized

$$\varepsilon(i) = \frac{1}{2}\eta_{ab}^{-1}\left(\frac{\phi_a(i+1) - \phi_a(i-1)}{2h}\right)\left(\frac{\phi_b(i+1) - \phi_b(i-1)}{2h}\right) + U(\phi(i)) \tag{4.10}$$

where we used a central difference

$$\frac{dy}{dx}(i) = \frac{y(i+1) - y(i-1)}{2h} \tag{4.11}$$

in order to reduce the numerical error in the caculation of the derivative. When using this, we have to take particular care about the boundaries of our interval. Notice that, when trying to use equation (4.10) to calculate the energy at indices $i = 0$ or $i = N - 1$, we would need the values $\phi_a(-1)$ and $\phi_a(N)$, which are not defined. At the boundaries, then, we need to use a forward (backward) difference when discretizing derivatives at $i = 0$ ($i = N - 1$). We can use this approach to compute any quantities we are interested in that depend on the fields or their derivatives – always taking care at the boundaries.

The total static energy

$$E = \int_{-\infty}^{\infty} dx\ \varepsilon \tag{4.12}$$

can also be calculated. Since we are studying configurations with localized energy density, the integral can be truncated at the boundaries. A simple way to integrate numericaly – and the one we used here – is

$$E = \sum_{i=0}^{N-1} h\varepsilon(i). \tag{4.13}$$

One other thing we can do is the construction of multi-soliton configurations. This kind of solution, which asymptotically consist of well separated "single"-solitons, will be used in the next section as initial conditions to the simulation of the scattering processes. For simplicity, consider a theory of a single field ϕ and three consecutive vacua points $(\phi_{0,1}, \phi_{0,2}, \phi_{0,3})$. The configuration we want to build will interpolate $\phi_{0,1}$ and $\phi_{0,2}$ and then $\phi_{0,2}$ and $\phi_{0,3}$. If $\phi_{0,1} \neq \phi_{0,3}$ then the configuration is called a kink-kink or 2-kink. Another interesting case arises when $\phi_{0,1} = \phi_{0,3}$, that is, the configuration starts at $\phi_{0,1}$, goes to $\phi_{0,2}$ and returns to $\phi_{0,1}$. This is what is called a kink-antikink configuration. Notice that it belongs to the same homotopy class as the vacuum configuration $[\{\phi_{0,1}, \phi_{0,1}\}]$, *i.e.*, topology do not forbid it to decay in a vaccum configuration under time evolution, and that is exactly what we will observe in the scattering of a kink-antikink system where these objects move towards each other with low relative speed.

The procedure to construct this type os configuration is as follows. We start by solving the system (4.6) for the first solution ϕ_1, which intorpolates $\phi_{0,1}$ and $\phi_{0,2}$. We want it to be in the first half of the interval, so we choose $i = N/4$ as the starting index, and as initial condition we choose halfway between the vacua, that is,

$$\phi_1\left(\frac{N}{4}\right) = \frac{\phi_{0,2} - \phi_{0,1}}{2}. \tag{4.14}$$

The second soliton ϕ_2 will be centered at the second half of the interval and will interpolate

$\phi_{0,2}$ and $\phi_{0,3}$, *i.e.*, the initial value will be

$$\phi_2\left(\frac{3N}{4}\right) = \frac{\phi_{0,3} - \phi_{0,2}}{2}. \tag{4.15}$$

Having integrated (4.6) for both ϕ_1 and ϕ_2, we have two 1-soliton solutions. In order to obtain a superposition of both, we simply sum them. This sum will be shifted by a factor $\phi_{0,2}$, as can be seen in the limits

$$\phi(-\infty) = \phi_1(-\infty) + \phi_2(-\infty) = \phi_{0,1} + \phi_{0,2} \tag{4.16}$$

and

$$\phi(\infty) = \phi_1(\infty) + \phi_2(\infty) = \phi_{0,2} + \phi_{0,3}, \tag{4.17}$$

so we correct it by subtracting this same factor. Thus the initial multi-soliton configuration can be achieved by

$$\phi = \phi_1 + \phi_2 - \phi_{0,2}. \tag{4.18}$$

As an example, let us consider the sine-Gordon model. Let ϕ_1 be a kink centered at $-x_0$ that interpolates 0 and 2π

$$\phi_1 = 4\arctan(e^{x+x_0}) \tag{4.19}$$

and ϕ_2, a kink solution centered at x_0 interpolating 2π and 4π

$$\phi_2 = 4\arctan(e^{x-x_0}) + 2\pi. \tag{4.20}$$

A 2-kink solution can be constructed by taking

$$\phi = \phi_1 + \phi_2 - 2\pi = 4\arctan(e^{x+x_0}) + 4\arctan(e^{x-x_0}) \tag{4.21}$$

if the distance $2x_0$ between the centre of these kinks is large enough. The process of construction of this configuration is shown in figure (4.1). The equations we are interested in are not linear, and therefore the sum of solutions is not, in general, a solution. But, numericaly, we do not need it to be an exact solution; it can be **almost** a solution, that is, a solution up to a target numerical accuracy.

If we have a differential equation

$$\mathcal{D}(\phi) = 0 \tag{4.22}$$

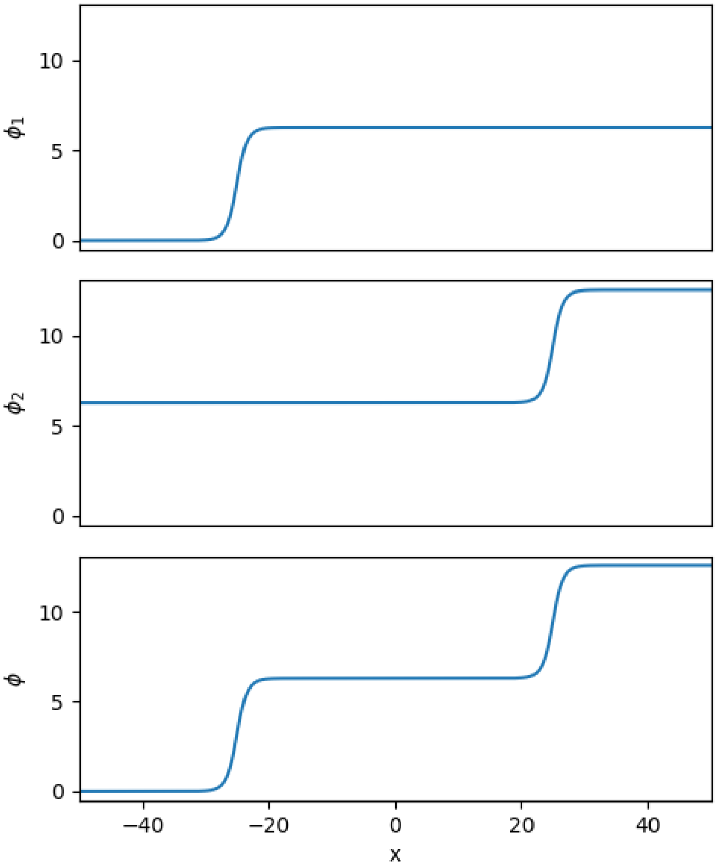


Figure 4.1: Numerical construction of a 2-kink configuration of the sine-Gordon model. The top plot is a kink cetered at $x = -25$. The middle one is a kink centered at $x = 25$. And the bottom plot is the sum of the two configurations above.

where $\mathcal{D}$ is a differential operator, we can define a variable

$$eq(x) \equiv \mathcal{D}(\phi(x)) \tag{4.23}$$

that takes the value of the differential operator $\mathcal{D}$ applied at a configuration $\phi(x)$ at each point x. If ϕ is a solution to (4.22), eq would vanish at each x. Numerically, then, we could use eq to test how accurately a given configuration satisfies a certain differential equation. In the case of the sine-Gordon model, for example, we can define eq to be the static Euler-Lagrange equation of the model

$$eq = \frac{d^2\phi}{dx^2} - \sin(\phi) \tag{4.24}$$

and test it for the configuration (4.21). Figure (4.2) shows the result of this calculation. As it can be seen, the accuracy is up to order 10^{-5} when using a step $h = 0.005$, which is a good approximation. And it can be made even smaller, by change of the step size h. As a rule of thumb, we can sum the 1-soliton solutions to get a multi-soliton system when the individual

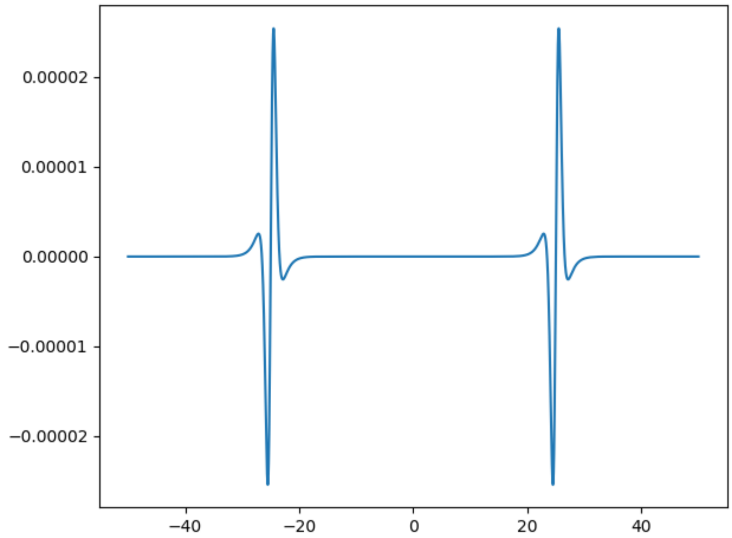


Figure 4.2: Numerical test of the satisfiability of the sine-Gordon static equation by a 2-kink configuration

ones are considerably separated.

Here the interpretation of a static configuration being the trajectory of a particle parametrized by x comes in handy. The particle starts at $x = -\infty$ in vaccum $\phi_{0,1}$ and somewhere along the way, say around $x = -x_0$, it tunnels to $\phi_{0,2}$, where it would arrive only in $x = \infty$. But if $x_0 \gg 0$, this means that at $x = 0$ it would be very close to $\phi_{0,2}$. For our purposes, that could be close enough so that we consider that it in fact arrived there. From vacua $\phi_{0,2}$ at $x = 0$ the particle goes on its path until around $x = x_0$, when it tunnels to vacuum $\phi_{0,3}$. This way we can see that it is possible to construct an approximate solution by summing two solutions in the limit that they are far apart.

4.2 Dynamical equations

The full equations of motion of the theories we are interested in solving are non-linear wave equations of general form

$$\partial_{tt}\phi - \partial_{xx}\phi = F(\phi, \partial\phi, t, x) \tag{4.25}$$

where $\partial\phi$ denotes any first derivative terms F might contain. We can transform this second order equation into a set of first order equations (on the time parameter) by defining

$$\begin{aligned} \partial_t \pi &= \partial_{xx}\phi + F \\ \partial_t \phi &= \pi. \end{aligned} \tag{4.26}$$

We, then, discretize the parameters (time and space coordinates) into vectors of components $t(i)$ and $x(j)$ and step sizes Δt and Δx, respectivelly. The fields $\phi(t, x)$ and $\pi(t, x)$ become matrices

$$\begin{aligned} \phi(i,j) &\equiv \phi(t(i), x(j)) \\ \pi(i,j) &\equiv \pi(t(i), x(j)) \end{aligned} \tag{4.27}$$

and equations (4.25) become a system of recurrence relations, when discretized

$$\begin{aligned} \pi(i+1,j) &= \pi(i,j) + \Delta t \left(\frac{\phi(i,j+1) - 2\phi(i,j) + \phi(i,j-1)}{\Delta x} + F(i) \right) \\ \phi(i+1,j) &= \phi(i,j) + \Delta t \; \pi(i,j), \end{aligned} \tag{4.28}$$

where we have used a forward difference (4.3) in order to discretize the first derivatives in time and a central difference

$$\frac{d^2 y}{dx^2}(i) = \frac{y(i+1) - 2y(i) + y(i-1)}{h} \tag{4.29}$$

in the second order space derivative.

Given the initial values $\phi(0,j)$ and $\pi(0,j)$, *i.e.*, the first lines of the field matrices, we can use the recurrence relations (4.28) to iteratively integrate the equations of motion. Here we also have to be careful about the spatial boundaries when discretizing the derivatives that appear in the r.h.s. of (4.28), so that we do not try to use indices that are not available. For second derivatives, the forward difference is given by

$$\frac{d^2 y}{dx}(i) = \frac{y(i+2) - 2y(i+1) + y(i)}{h} \tag{4.30}$$

and the backward difference is

$$\frac{d^2 y}{dx}(i) = \frac{y(i-2) - 2y(i-1) + y(i)}{h}. \tag{4.31}$$

When we have analytic static solutions for the field $\phi(x)$ we can always make a Lorentz boost $x \mapsto \gamma(x - vt)$ in order to obtain a moving configuration. In the case of multi-soliton solutions we boost the single soliton solutions ϕ_1 and ϕ_2, and choose opposite speeds for each, since we want to study their scattering. Then we can construct the system configuration as in

$$\phi(t,x) = \phi_1(\gamma(x + x_0 - vt)) + \phi_2(\gamma(x - x_0 + vt)) - \phi_{0,2} \tag{4.32}$$

where we have chosen ϕ_1 to be centered at $-x_0$ and moving to the right with speed v, and ϕ_2 to be centered at x_0 and moving to the left with speed v [2]. We can, then, take the partial time derivative of (4.32) to determine $\pi(t,x)$ and, with this and (4.32), it is possible to calculate the

[2] Notice, though, that (4.32) is **not** a solution – or even an approximate solution – to the field equations for every t. This equation is only to be used in the construction of the initial data for the simulation

initial data $\phi(0, j)$ and $\pi(0, j)$. This is all information needed to perform the computation of (4.28).

A slight setback arises when we are dealing with theories which do not have analytical static solutions. Fortunatelly, this is not too difficult to overcome. If the theory has BPS equations we can use the methods decribed in the last section to construct a initial field configurarion. If not, there are other methods that can be used, *e.g.*, the relaxation or the shooting methods – these will not be presented here, though, but literature is vast on the subject [11].

The aforementioned setback concerns the Lorentz boost and later computation of the field π, since we have no means to do so in the usual sense as in the analytic case. These two problems are solved at once with one simple trick. Supose we have a static configuration $\phi(x')$ relative to reference frame S'. A Lorentz boost is then a change of coordinate system from S' to S by $x' = \gamma(x - vt)$. The time derivative in S can be performed implicitly

$$\partial_t \phi(x') = \frac{\partial x'}{\partial t}\frac{d\phi}{dx'} \tag{4.33}$$

$$= -v\gamma \frac{d\phi}{dx'} \tag{4.34}$$

and the information about the speed at which the configuration is moving is already encoded in this calculation. Now, having a static configuration – even a numericaly calculated one – we have the means to boost it and simulate scattering processes. A important remark is that when we calculate the initial configuration, we are in the static reference frame with coordinate x'. In order to do the simulation, we should rescale the coordinate according to the Lorentz transformation $x = x'/\gamma$ and also the grid step $\Delta x = \Delta x'/\gamma$.

In order to build these moving solutions we have two obvious approaches: we can boost the separate solitons ϕ_1 and ϕ_2 and then sum them and their derivatives π_i, or we can boost the full configuration ϕ altogether. Nonetheless, in order to do so, we have to be careful to boost each half independenty. That is, for $i = 0$ until $i = N/2$, we use equation (4.33) with speed v in order to boost the leftmost soliton; and for $i = N/2$ until $i = N - 1$ we use (4.33) with opposite sign speed $-v$, so that the rightmost soliton moves in the opposite direction.

The two approaches give basically the same result, so it is merely a matter of taste which one to use. In the simulations presented in the next section, we have used the second one, particularly.

4.3 Simulation of soliton scattering in a $\mathfrak{su}(2)$ FKZ model

In section 3.2.1 we presented the $\mathfrak{su}(2)$ FKZ models, constructed following the generalized BPS formalism. We have then simulated several scattering processes of the solutions found in the specific model constructed from the $2 \oplus 3$ representation. The dynamical Euler-

Lagrange equation for this model is

$$\partial_{tt}\phi - \partial_{xx}\phi + V_{\phi} = 0 \tag{4.35}$$

where

$$V_{\phi} \equiv \frac{dV}{d\phi} = -b\sin^3\phi\left(1 + b\cos\phi\right) + \frac{1}{2}\sin(2\phi)\left(1 + b\cos\phi\right)^2. \tag{4.36}$$

So the system of first order equations we have to solve is

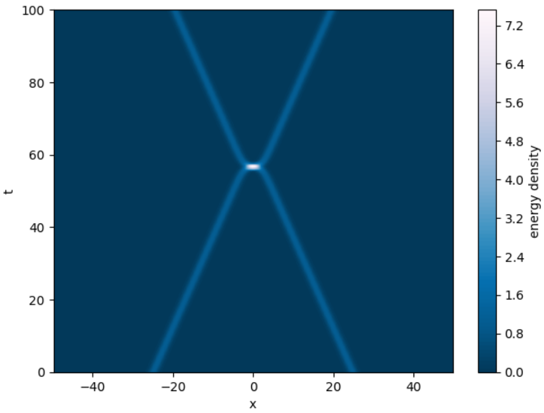
$$\begin{aligned}\partial_t\pi &= \partial_{xx}\phi - V_{\phi}\\ \partial_t\phi &= \pi\end{aligned} \tag{4.37}$$

which after discretized reads

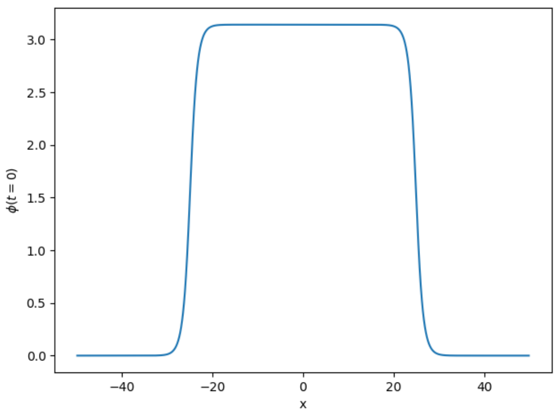
$$\begin{aligned}\pi(i+1,j) &= \pi(i,j) + \Delta t\,\left(\partial_{xx}\phi(i,j) - V_{\phi}(i,j)\right)\\ \phi(i+1,j) &= \phi(i,j) + \Delta t\,\pi(i,j).\end{aligned} \tag{4.38}$$

We simulated the scattering of kink-kink and kink-antikink systems for several values of the parameter b and speed v. The initial configuration was always constructed having the 1-soliton solutions equally apart from the origin of x' and boosted inwards with speeds of equal modulus v. The grid steps were constant and, for the results presented here, we used $\Delta x' = 0.005$ for the calculation of the initial configuration; $\Delta x = \Delta x'/\gamma$ and $\Delta t = \Delta x/2$ were used in the computation of the scattering processes. We present some results that summarize the general properties of these processes in figures 4.3 through 4.5.

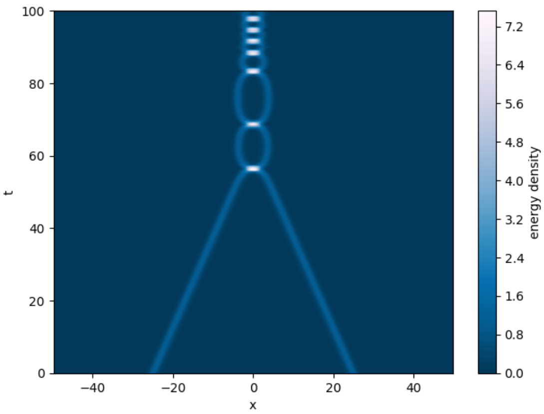
In figure 4.3 we have scatterings of low initial speed $v = 0.1$ for values of the parameter $b = 0.0$, $b = 0.1$ and $b = 2.0$. It can be seen that in the case $b = 0.0$, no bound state was formed. This is due to the fact that, for this value of b, the theory is equivalent to a sine-Gordon model, which is integrable. In integrable theories, a infinite number of conserved charges stabilizes the configuration and preserve the individual structure of the solitons. As it can be observed in figures 4.3c and 4.3e, this was not the case for $b > 0$, where we had the formation of bound states. This confirms that this model is not integrable for $|b| > 0$. Moreover, as kink-antikink configurations are in the same homotopy class of the vacuum, they can decay dynamically. This is exactly what we observe here, as these systems tend to emit radiation and dissipate slowly after forming a bound state.



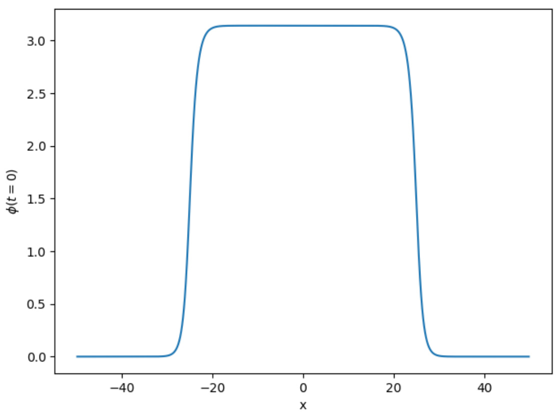
(a) Kink-antikink scattering for $b = 0.0$ and $v = 0.1$.



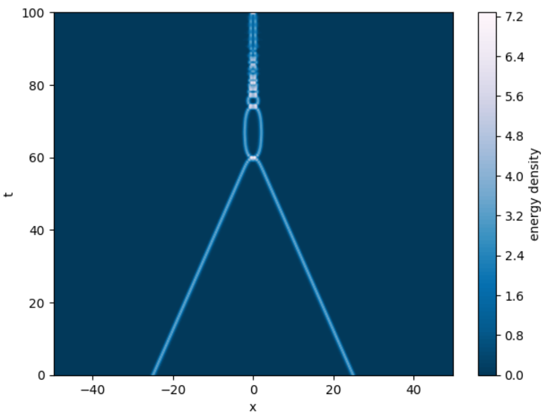
(b) Initial profile for scattering in (a).



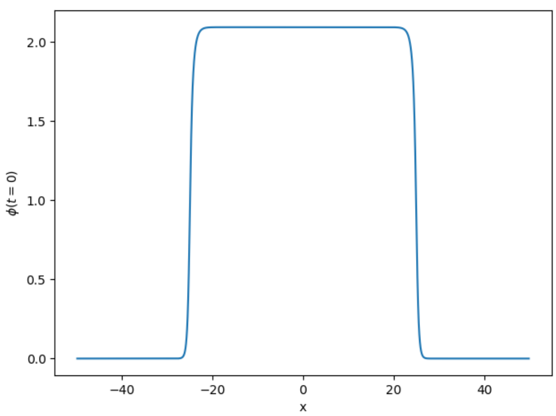
(c) Kink-antikink scattering for $b = 0.1$ and $v = 0.1$.



(d) Initial profile for scattering in (c).



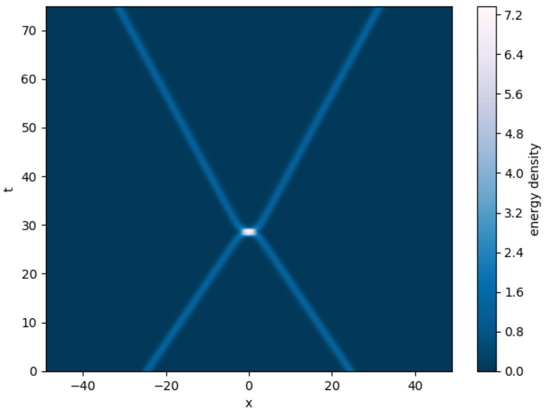
(e) Kink-antikink scattering for $b = 2.0$ and $v = 0.1$.



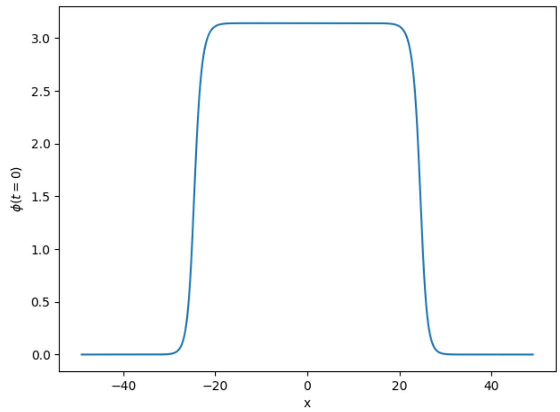
(f) Initial profile for scattering in (e).

Figure 4.3: Simulation of scattering of kink-antikink systems of the $\mathfrak{su}(2)$ FKZ model.

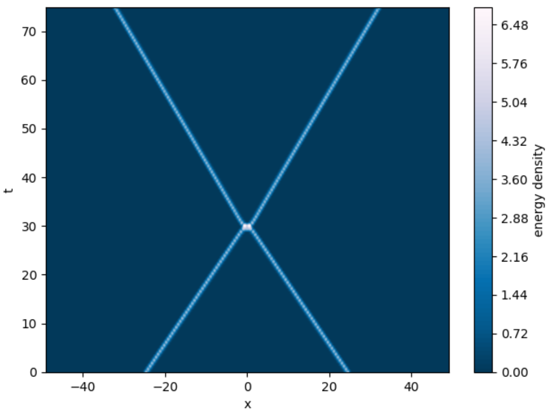
In figure 4.4 we present the scattering of kink-antikink systems for $b = 0.1$ and $b = 2.0$ at a slightly greater initial speed $v = 0.2$. Here we did not see the formation of bound states. The existence of a critical speed above which we do not observe bound states is expected, even in non-integrable theories. However, we were surprised by the fact that these models presented such a low critical speed. In the ϕ^4 model [12], for example, the critical speed is about $v = 0.4$.



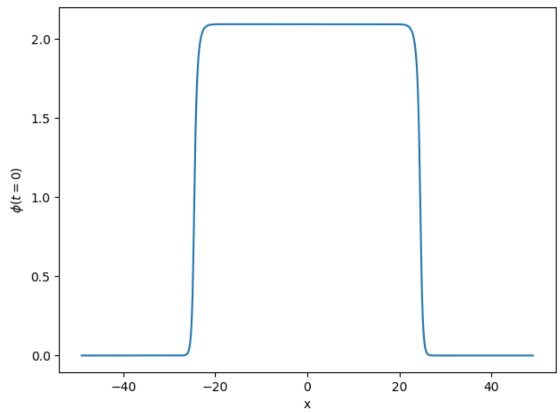
(a) Kink-antikink scattering for $b = 0.1$ and $v = 0.2$.



(b) Initial profile for scattering in (a).



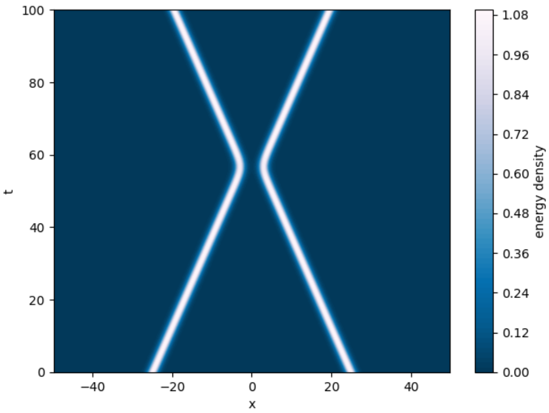
(c) Kink-antikink scattering for $b = 2.0$ and $v = 0.2$.



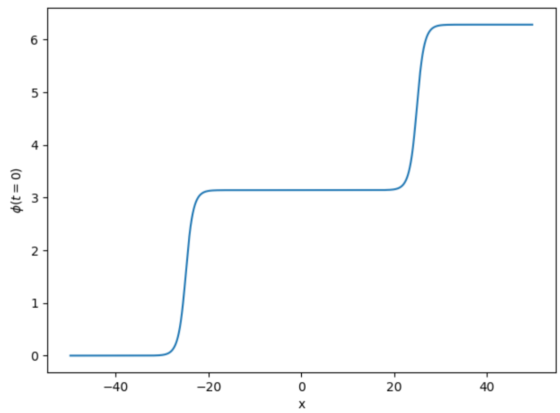
(d) Initial profile for scattering in (c).

Figure 4.4: Simulation of scattering of kink-antikink systems of the $\mathfrak{su}(2)$ FKZ model.

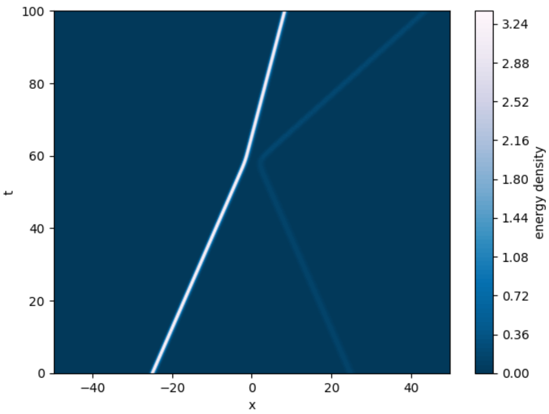
In figure 4.5 we have the only kink-kink scatterings we will present. As 2-kink configurations are in homotopy classes different from the vacuum, topology prevents them from decaying. So in these cases we cannot have the formation of a bound state and the scattering processes can at most be inelastic for non-integrable theories, where we have emission of a bit of radiation. Figure 4.5a shows the scattering of the integrable case $b = 0$. Notice that the solitons do not touch each other, being repelled before it happens. In figure 4.5c we have a very interesting case for $b = 2.0$. Because the vacua are not equally spaced $|b| > 1$, we have the existence of kinks with different sizes, which confers them different energies. So the scattering is not symmetric and we have that the "big" kink is a bit slowed down, but not completely reflected; while the "little" kink accelerates considerably after the collision.



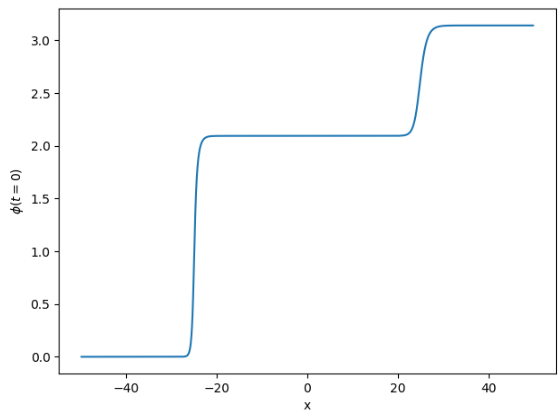
(a) Kink-kink scattering for $b = 0.0$ and $v = 0.1$.



(b) Initial profile for scattering in (a).



(c) Kink-kink scattering for $b = 2.0$ and $v = 0.1$.



(d) Initial profile for scattering in (c).

Figure 4.5: Simulation of scattering of kink-kink systems of the $\mathfrak{su}(2)$ FKZ model.

A minimal amount of radiation was observed after the collision in each simulation with $b > 0$, when compared with other non-integrable theories, such as the ϕ^4 model [12]. This could be a hint that these models present some quasi-integrable [13] character. Further, it is necessary to determine the precise critical speed for the cases $b > 0$ and to test for the existence of “bounce windows”[3]. We are currently working on both of these last tasks.

[3]Specific speeds below the critical for which we do not observe the formation of a bound state.

Chapter 5

Concluding remarks

The main purpose of this dissertation was twofold: **i)** the application of the generalized BPS formalism in the construction of the $\mathfrak{g}_2$ and $\mathfrak{su}(4)$ FKZ models; and **ii)** the simulation of 2-soliton configurations in a $\mathfrak{su}(2)$ FKZ model.

In chapter 2 we reviewed some concepts that are essential to the general study of topological solitons. In particular, we reviewed basic definitions of topology, aimming in the understanding of homotopy groups and how we can use them to classify our solutions. Then we presented a derivation of Derrick's theorem for scalar fields, justifying our choice to work in $(1 + 1)$-dimensional spacetime. Finally, we presented a general approach for the construction of the BPS equations for a single scalar field and gave two canonical examples, namely, the ϕ^4 and sine-Gordon models.

Chapter 3 dealt with the generalization of the BPS equations for a multiplet of fields. We reviewed the original construction of the equations from the topological charge and energy functionals and how we could use it to define new theories from Lie algebras, the FKZ models. In particular, we reviwed the models based on algebra $\mathfrak{su}(2)$ and successfully constructed models based on algebras $\mathfrak{g}_2$ and $\mathfrak{su}(4)$, cases that were not considered in the original paper [5]. A thorough description of the vacuum structure of the $\mathfrak{su}(4)$ is still needed. Further, the stability analysis for the static solutions of both $\mathfrak{g}_2$ and $\mathfrak{su}(4)$ models is pending and we are currently working on this project.

In chapter 4 we reviewed some numerical methods for the construction of static configurations from the BPS equations and further simulation of the dynamics of this objects, when forced to scatter. We, then, presented some scattering processes of a $\mathfrak{su}(2)$ FKZ model. The simulations revealed that this model is indeed non-integrable. Nonetheless, the minimal amount of radiation emitted in the interactions were not expected, which could indicate a quasi-integrable aspect of the model. Moreover, we are currently working on the precise determination of the critical speeds and testing for the existence of bounce windows in this model.

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