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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in the area of Astrophysics, Cosmology and Gravitation, in the research line of Cosmology and Quantum Field Theory.

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Dedicado a la memoria de mi tia Soraida Agudelo. Por siempre viva en mi memoria y mi corazón.

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Abstract

The core of the present thesis is the possibility of the change with the energy scale (running) of the cosmological constant. Theoretically, this running in the IR region is not ruled out. On the other hand, from the Quantum Field Theory (QFT) viewpoint, the energy released due to the variation of the cosmological constant in the late Universe cannot go to the matter sector. For this reason, the phenomenological bounds on such a running are not sufficiently restrictive. The situation can be different in the early Universe when the gravitational field was sufficiently strong to provide an efficient creation of particles from the vacuum. We develop a framework for systematically exploring this possibility. It is supposed that the running occurs in the epoch when the Dark Matter (DM) already decoupled and is expanding adiabatically, while the usual matter should be regarded approximately massless and can be abundantly created from vacuum due to the decay of vacuum energy. By using the handy model of Reduced Relativistic Gas (RRG) for describing the Warm Dark Matter (WDM), we consider the dynamics of both cosmic background and linear perturbations and evaluate the impact of the vacuum decay on the matter power spectrum and to the first CMB peak. Additionally, using the combined SNIa+BAO data, we find the best-fit values for the free parameters of the model.

Additionally, it is known than the inclusion of spatial curvature can modify the evolution of matter perturbations and affect the Large Scale Structure (LSS) formation. We quantify the effects of the non-zero space curvature in terms of LSS formation for a cosmological model with a RCC and a WDM component. The evolution of density perturbations and the modified shape of its power spectrum are also reconstructed and analyzed in this context.

Finally, it is analytically constructed the scalar field actions minimally and non-minimally coupled to gravity, which are equivalent to RRG (describing the WDM component) in the sense they produce the same cosmological solutions for the conformal factor of the metric. In particular, we construct the scalar theory which corresponds to the model of an ultra-relativistic ideal gas of spinless particles possessing conformal symmetry. The possibility of supplementing our scalar field model for WDM with dynamical dark energy in the form of a RCC is also considered.

Keywords: Running cosmological constant, Running vacuum energy, warm dark matter, observational constraints, reduced relativistic gas, scalar field theory.

Resumo

O núcleo da presente tese é a possibilidade da variação da constante cosmológica com a escala de energia. Teoricamente, essa variação na região IR é um modelo possível e viável. Porém, do ponto de vista da Teoria Quântica dos Campos (QFT), a energia resultante dessa variação da constante cosmológica não pode ser absorbida pelo setor de matéria. É por isto que, os limites fenomenológicos sobre tal variação não são suficientemente restritos. Essa situação pode ser diferente no universo primordial, época na qual o campo gravitacional foi grande o suficiente para facilitar uma criação eficiente de partículas a partir do vácuo. Desenvolvemos aqui um marco teórico para explorar esta possibilidade de forma sistemática, sob o suposto de que dita variação ocorre numa época na qual a matéria escura (DM) está desacoplada e se expande de forma adiabática, enquanto que a matéria usual pode ser considerada como sem massa e pode ser criada de forma abundante a partir do decaimento da energia do vácuo. Mediante o uso do simples modelo de gás relativistico reduzido (RRG) para descrever matéria escura morna (WDM), consideramos aqui a dinâmica tanto no nível de fundo quanto no nível perturbativo e avaliamos o impacto desse decaimento do vácuo no espectro de potências da matéria e no primeiro pico acústico do CMB. Adicionalmente, usando dados combinados de SNIa+BAO, encontramos os valores do melhor ajuste para os parâmetros livres do nosso modelo.

Adicionalmente, é bem conhecido que a inclusão da curvatura espacial pode modificar a evolução das perturbações da matéria e afetar a formação de estruturas a grande escala (LSS). Quantificamos os efeitos dessa curvatura não nula em termos da LSS para um modelo cosmológico considerando uma RCC e WDM. A evolução das perturbações de densidade e a forma modificada do espectro de potências é também reconstruída neste contexto.

Finalmente, as ações de campo escalar mínima e não-mínima equivalentes ao RRG (descrevendo WDM) são construídas analíticamente, onde tais ações e suas equações de movimento, reproduzem as mesmas soluções para o fator de escala cosmológico presente na métrica FRLW. Em particular, construímos a teoría escalar correspondente com o modelo de gás ideal ultra-relativistico de partículas com spin nulo que possui simetria conforme. É considerada também a possibilidade de complementar nossa descrição escalar para WDM com um componente de energia escura (DE) dinâmica na forma de uma constante cosmológica variável (RCC).

Palavras chave: Constante cosmológica variável, energia de vácuo variável, matéria

escura morna, gás relativistico reduzido, teoría de campo escalar.

Introduction

The standard model of cosmology (ΛCDM) describes the properties and evolution of the universe as a whole, considering the existence of a new kind of cosmic fluids known as dark energy (DE) and dark matter (DM) and whose theoretical predictions are in very good agreement with observational data [4, 5]. However, although this model seems to pass most of the experimental tests, there are still some discrepancies and tensions with data that cannot be explained naturally within this framework [6].

In the ΛCDM model, the role of DE is assumed by the positive cosmological constant (CC) Λ, which is considered as a fluid with negative pressure, as the most natural and simple explanation for the current accelerating phase [7]-[8]. Nonetheless, this leads to the well known CC problem, opening new searches and possibilities for the solution of the DE problem [9, 10]. These new developments have also given rise to some extensions of the standard model or even modifications of the gravitational theory which is based on [11, 12].

The improving quality of the data of observational cosmology leads to better estimates of the equation of state of the Dark Energy, which is driving the accelerated expansion of the Universe. The current data are consistent with the value of $w = -1$, which means the cosmological constant. From the quantum field theory point of view, the cosmological constant is a necessary element of a consistent semiclassical theory [13][14][15][16] and hence it should not be taken as a surprise that it is nonzero.

The ultimate word about the origin of the Dark Energy belongs to observations. It can not be ruled out that at some moment the analysis of the data proves that the density of the Dark Energy changes with time. Does this mean that there is another component of the Dark Energy, besides the cosmological constant? Before answering this question, one has to understand whether the cosmological constant can be not exactly a constant. It is a standard assumption that the observable density of the vacuum energy is a sum of the vacuum counterpart and the contribution generated by a symmetry breaking, e.g. at the electroweak and QCD scales. In principle, both vacuum and induced parts can be variable due to quantum effects.

The variation of cosmological “constant” term, because of the quantum effects, can be explored employing the renormalization group running of this parameter [17][18]. The simplest version of such a running can be described in the framework of a minimal subtraction scheme in curved space [14][19] (see also [15]), but this kind of running leads to the inconsistent cosmological model [17]. The standard interpretation is that the "correct" running at low energies (in the IR) should take into account the decoupling of the massive fields. Such decoupling cannot be verified for the cosmological constant case [20], but the non-running can be proved neither [18]. Thus, the situation is such that one can explore the running cosmological constant only in the phenomenological setting. However, it is important to have this setting well-defined. And in this respect, the main point is what happens with the energy when the cosmological constant varies according to the evolution of the Universe and the corresponding change of the energy scale.

It is well-known that the quantum or semiclassical corrections to the action of gravity are typically non-local and rather complicated (see e.g. [20]). However, one can identify the terms responsible for the running of the cosmological constant using the global scaling arguments [16]. Starting from this point, one can meet two distinct possibilities to implement the cosmological constant running in cosmology.

The first one assumes the energy exchange between vacuum and matter sectors. The cosmological model that emerges from this assumption has essential technical advantages. In particular, the evolution of the cosmological background can be easily described using elementary functions [21] and the analysis of perturbations is also relatively simple [1]. For this reason, this model became popular (see, e.g, the review [10] and the recent publication [22][23]), regardless of the existing conceptual difficulties, that will be described below.

The second model is much more consistent for the low-energy regime, it is based on the conservation law not involving the matter sector, and assumes a mixture between the cosmological constant term and the Einstein-Hilbert action, that means a running of the Newton constant G. This model is more complicated technically, and also the phenomenological restrictions on the unique free parameter ν are very weak, at least from the analysis of structure formation [24][1]. In this thesis, we shall concentrate on the models of running cosmological constant of the first kind and explore the physical conditions where this model makes sense.

On the other hand, with respect to the dark matter, we know that describing DM component in terms of particles or fields also represents an open question. Besides the theoretical and experimental difficulties in detecting DM, there is a possibility to assume that DM is warm (WDM) instead of cold (CDM). It is known that the relativistic warmness of DM can change the global dynamic of the universe and provide certain phenomenological advantages [27][28][29]. The full standard description of WDM implies the use of the Boltzmann equation. However, the problem can be greatly simplified using the reduced relativistic gas model (RRG). Historically the first use of the equation of state equivalent to RRG was in the

[1]In compensation, running G has interesting astrophysical applications (see e.g. [25][26]).

pioneering work of A.D. Sakharov [30] on the matter acoustic oscillations. Later on, it was rediscovered in [2] as a simplification to the relativistic Maxwell distribution and the cosmological model constructed on this base.

The main point of the RRG is that it assumes the same kinetic energy for all particles of relativistic gas. Such an artificial ideal relativistic gas model has a very simple equation of state with the unique free parameter b, characterizing its warmness. On the other hand, this equation of state closely reproduce similar equation in the Jüttner model [31], based on the relativistic Maxwell distribution. These two features enable one to use RRG, e.g., for the simplified phenomenological description of WDM [2][32][33].

The simplicity of RRG is especially welcome in the theories with technical and conceptual complications, such as the cosmological models with running parameters. As far as most interesting models of running ρ_Λ (see e.g. [21] and [1]) are consistently applicable only at high energy scale, we need to formulate them in the framework of early universe, when the DM is supposed to have more warmness than today. Then the RRG becomes a useful tool that enables to get the main features of the model with the reduced amount of numerical calculations and more clear physical understanding of the results. For this reason, in the recent work [34] we have started the exploration of the running ρ_Λ and RRG model for the early Universe.

Different from the later epochs, in the epoch soon after inflation, the creation of at least the Standard Model particles from the vacuum (see e.g. [35], [36] and references therein), is not suppressed by the low energy density of the gravitational field of the cosmological background. At the same time, the running of the cosmological constant density in the high-energy regime is the phenomena which may leave observational traces in the late universe. This is the subject of the study in Ref. [34]. We make the next step and quantify the effects of the non-zero spatial curvature in the model with running ρ_Λ and the WDM contents described by RRG model in the early universe. Our purpose is to evaluate the effect of spatial curvature on some observables in the context of LSS formation as the matter power spectrum. Technically, our purpose is to evaluate the constraints on the free parameters ν and b in the presence of Ω_k^0, using SNIa and DR11 cosmic data-sets [37][5].

Indeed, it is interesting to include curvature in the model of Ref. [34], and not only for the sake of generality. In the last years, there was an intensive discussion of the observational constraints on the space geometry, including the curvature of the universe. For example, some observational results including SNIa, H, BAO, QSO, etc have shown statistical consistency with a closed curvature universe (see e.g. the references [38]-[39]). Thus, it looks natural to include consideration of space curvature in the model with the running cosmological constant.

Another common possibility for explaining the current and primordial properties of an expanding universe, it can be given in terms of a scalar field, where this field can be minimally or non-minimally coupled to gravity, it has a time-dependent equation of state and its dynamical evolution can be compatible with the solution of the DE problem and even it can provide a suitable inflationary dynamics with excellent confidence level according to observations [40]-[41].

It is interesting then, reconstructing and discussing a consistent scalar field theory for describing WDM using the useful approximation of the simple RRG model, as well as its cosmological consequences. For this purpose, we consider the description in terms of a minimal and non-minimal scalar field using the hydro-dynamical approach and the properties of the conformal transformation. Additionally, the possible inclusion of a dynamical dark energy component in the form of an RCC within this scalar field description is then considered.

This thesis is organized as follows. In Chapter 1 we give a brief review of some essential concepts about modern relativistic cosmology, which are going to be used along this thesis. In Chapters 2 and 3, we mention the main properties and some applications of the running vacuum energy from quantum effects and the simple warm dark matter model using the reduced relativistic gas. In chapter 4 we consider both of mentioned models evolving together from a primordial phase of the universe. In Chapter 5, we consider the inclusion of spatial curvature and its effect on the matter power spectrum and the new possible constraints for the free paramaters of the joint model, this is, the running parameter and the warmness. In Chapter 6 we present an scalar field theory for the warm dark matter component as treated as a reduced relativistic gas, also exploring the possibility of the inclusion of a dynamical dark energy component in a form a running vacuum energy. Finally, in Chapter 6.3 we draw our conclusions, perspectives and summarize our contributions.

Chapter 1

Basics on modern cosmology

In this chapter we shall summarize some general and necessary concepts and quantities that will be used along this document without discussing them in much details and following the references [42] [43][44].

1.1 Elements of relativistic cosmology

1.1.1 Einstein field equations

Gravity is one of four fundamental interactions in nature. It is described by GR and is the dominant interaction on very large scales. In the framework of this theory of gravitation, the dynamics of matter and the spacetime itself is governed by Einstein field equations (EFE), which are obtained by varying the total action [42]

$$S = S_{EH} + S_M,$$

with respect to the metric tensor $g^{\mu\nu}$ to get

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{1.1}$$

which relate the matter content of the universe with the gravitational field described in terms of its geometrical properties and where

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\,(R - 2\Lambda), \qquad S_M = \int d^4x \sqrt{-g}\, \mathcal{L}_M \tag{1.2}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{1.3}$$

are the Einstein-Hilbert action, the matter action and the definition of the energy-momentum tensor, respectively. This $T_{\mu\nu}$ concentrates all possible matter components such as baryons, photons, neutrinos or even DE or DM. Here, $\kappa^2 = 8\pi G$ and Λ is the famous cosmological constant firstly introduced by Einstein in order to describe an static universe, although nowadays is associated to the dark energy (DE). In this form Λ has a geometric interpretation, but it is also possible to include it through a fluid with density and pressure given by

$$\rho_\Lambda = \frac{\Lambda}{\kappa^2}, \qquad p_\Lambda = -\rho_\Lambda \tag{1.4}$$

such that we can write the EFE as

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \kappa^2 T^t_{\mu\nu}, \tag{1.5}$$

where the total energy-momentum tensor would be

$$T^t_{\mu\nu} = T^m_{\mu\nu} + T^\Lambda_{\mu\nu} \tag{1.6}$$

with

$$T^m_{\mu\nu} = (\rho_m + p_m) U_\mu U_\nu + p g_{\mu\nu}, \qquad T^\Lambda_{\mu\nu} = -\rho_\Lambda g_{\mu\nu} \tag{1.7}$$

so Λ can be interpreted as an additional matter fluid, whose energy density is associated to the vacuum energy.

On the other hand, the gravitational field in GR is characterized by the geometry of the cosmic manifold condensed in the metric tensor $g_{\mu\nu}$ and their related quantities such as the Riemann curvature tensor

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\nu \Gamma^\kappa{}_{\lambda\mu} - \partial_\mu \Gamma^\kappa{}_{\lambda\nu} + \Gamma^\rho{}_{\lambda\mu}\Gamma^\kappa{}_{\nu\rho} - \Gamma^\rho{}_{\lambda\nu}\Gamma^\kappa{}_{\mu\rho} \tag{1.8}$$

from which, contracting with the metric, it is possible to obtain

$$R_{\lambda\nu} = R^\kappa{}_{\lambda\kappa\nu} = g^{\kappa\gamma} R_{\gamma\lambda\kappa\nu}, \qquad R = R^\mu{}_\mu = g_{\mu\nu} R^{\mu\nu}.$$

known as Ricci tensor and curvature scalar. The quantities $\Gamma^\kappa{}_{\mu\nu}$ are the Christoffel symbols

$$\Gamma^\kappa{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left[\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right]. \tag{1.9}$$

As a consequence of Bianchi identities [42], we also have that the total energy-momentum tensor is a conserved quantity

$$\nabla^\mu T^t_{\mu\nu} = 0 \tag{1.10}$$

describing the evolution of the matter content in an expanding universe. It will be also useful to write the EFE in the alternative form

$$R_{\mu\nu} = \kappa^2 (T^t_{\mu\nu} - \frac{1}{2} T^t g_{\mu\nu}) \tag{1.11}$$

such that, where perturbations are considered, we can avoid the calculations related to curvature scalar R [45].

1.1.2 FRLW universe

Currently, we have strong evidence to accept that matter distribution in the universe, on very large scales, satisfies the cosmological principle (CP), so the universe on these scales is homogeneous and isotropic [44]. The solution to EFE that is compatible with this CP and an expanding universe is the Friedmann-Robertson-Lemaitre-Walker (FRLW) metric with maximally symmetric spatial part and whose line element reads

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2\right], \tag{1.12}$$

where $a(t)$ is the scale factor and k is the spatial curvature, whose values 1, 0 and -1, corresponds to spherical, plane and hyperbolic space, respectively. Considering an energy-momentum tensor for a perfect fluid as

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{1.13}$$

the EFE (1.1) for this FRLW metric take the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}, \qquad \left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.14}$$

known as Friedmann and Raychauduri equations. The conservation law (1.10) in this case yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{1.15}$$

such that we can apply this set o three equations for describing the dynamics and evolution of matter contents and scale factor in an homogeneous and isotropic background.

1.1.3 Cosmological parameters and the ΛCDM model

Let us recall some common definitions and main properties related to the standard cosmological model [44].

Relative density parameter

From the Friedmann equation we can define the critical energy density as the energy density such that $k = 0$, this is

$$\rho_c = \frac{3H^2}{8\pi G} \tag{1.16}$$

whose present value is

$$\rho^0{}_c = 1.878 \times 10^{-29}\, h^2\, g\, cm^{-3}. \tag{1.17}$$

It is also very useful to define the relative density parameter, which is defined as

$$\Omega = \frac{\rho}{\rho_c} \tag{1.18}$$

such that we are normalizing with respect to to the critical density. We can rewrite the Friedmann equation in the form

$$\Omega + \Omega_k = 1 \tag{1.19}$$

where the sum of all density parameters is always equal to unity and it has been defined the curvature density parameter

$$\Omega_k = \frac{\rho}{\rho_c}, \qquad \rho_k = -\frac{3k}{8\pi G a^2}. \tag{1.20}$$

In what follows, we shall use the also common normalization with respect to the critical density today

$$\Omega = \frac{\rho}{\rho^0{}_c} = \frac{8\pi G\rho}{3H_0^2} \tag{1.21}$$

so the Friedmann equation now takes the form

$$H^2 = H_0^2 \left(\sum_x \Omega^0{}_x f_x(a) + \frac{\Omega^0{}_k}{a^2} \right) \tag{1.22}$$

and we also have the closure relation

$$\sum_x \Omega^0{}_x + \Omega^0{}_k = 1 \tag{1.23}$$

where we have one $\Omega^0{}_x$ for each matter component.

Usual forms of matter

Assuming a matter component with equation of state $p = w\rho$, with w as a constant, the general solution of the cosmological continuity equation is given by

$$\rho = \rho_0\, a^{-3(1+w)} \tag{1.24}$$

such that in the case of $w = 0$ ($p = 0$), $w = 1/3$ ($p = \rho/3$) and $w = -1$ ($p = -\rho$), we are treating with cold and non-relativistic matter (dust-like or pressureless matter), hot and relativistic matter (radiation-like matter) and vacuum energy (Λ-like matter), respectively.

ΛCDM model

Also known as the concordance model, it is the most successful cosmological model in the light of all observations and is constructed on the basis of the CP and considering a total matter content composed by: Λ, cold dark matter (CDM), baryons and radiation (photons and neutrinos). The Hubble parameter for this case is given by

$$\left(\frac{H}{H_0}\right)^2 = \Omega_\Lambda + \frac{\Omega^0_m}{a^3} + \frac{\Omega^0_b}{a^4} + \frac{\Omega^0_r}{a^4} + \frac{\Omega^0_k}{a^2} \tag{1.25}$$

where according to recent observations we have the approximate values for the parameters

$$\Omega^0{}_\Lambda = 0.6911 \pm 0.0062, \qquad \Omega^0{}_m = 0.3089 \pm 0.0062 \tag{1.26}$$

with

$$\Omega^0{}_m = \Omega^0{}_m + \Omega^0{}_m, \tag{1.27}$$

including both CDM and baryons. However, we also have

$$\Omega^0{}_b h^2 = 0.02230 \pm 0.00014, \qquad \Omega^0{}_c h^2 = 0.1189 \pm 0.0010 \tag{1.28}$$

for $h = 0.68$ [46]. From this values we can conclude that our universe is composed by 69% Λ, 29% CDM and 5% baryons.

Perhaps this ΛCDM model and their related equations describe the global dynamics of the universe according to our observations, they are insufficient to understand the nature and fundamental properties of DM and DE, as well as the underlying mechanisms related to the large scale structure (LSS) or the properties of the cosmic microwave background (CMB) [43]. Therefore, it is needed to consider the presence of inhomogeneities introduced in the form of small perturbations as we briefly shall review in next section.

1.2 Introducing perturbations

The FRLW universe is only reliable on very large scales where the CP is valid, so if we want to understand the structure formation, which clearly represents a huge deviation from CP, we need to consider the evolution of the small perturbations in a flat expanding homogeneous and isotropic background [44][43].

Let us start defining the difference

$$\delta g_{\mu\nu}(x) = g_{\mu\nu}(x) - \bar{g}_{\mu\nu}(x) \tag{1.29}$$

where $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ are the background and the physical spacetime metrics, respectively. If we choose one gauge, such that [47]

$$|\bar{g}_{\mu\nu}| \gg \delta g_{\mu\nu} \tag{1.30}$$

we are dealing with perturbations and our physical spacetime would be composed by

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{1.31}$$

and consequently, for using this perturbed metric in the alternate form of EFE we would need to compute the quantities

$$\Gamma^{\rho}{}_{\mu\nu} = \bar{\Gamma}^{\rho}{}_{\mu\nu} + \delta\Gamma^{\rho}{}_{\mu\nu}, \qquad R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}, \tag{1.32}$$

where

$$\delta\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\lambda}\left[\partial_{\mu}\delta g_{\lambda\nu} + \partial_{\nu}\delta g_{\lambda\mu} - \partial_{\lambda}\delta g_{\mu\nu} - 2\delta g_{\lambda\alpha}\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right], \tag{1.33}$$

$$\delta R_{\mu\nu} = \partial_{\rho}\delta\Gamma^{\rho}{}_{\mu\nu} - \partial_{\nu}\delta\Gamma^{\rho}{}_{\mu\rho} + \bar{\Gamma}^{\rho}{}_{\mu\nu}\delta\Gamma^{\sigma}{}_{\rho\sigma} + \delta\Gamma^{\rho}{}_{\mu\nu}\bar{\Gamma}^{\sigma}{}_{\rho\sigma} - \bar{\Gamma}^{\rho}{}_{\mu\sigma}\delta\Gamma^{\sigma}{}_{\nu\rho} - \delta\Gamma^{\rho}{}_{\mu\sigma}\bar{\Gamma}^{\sigma}{}_{\nu\rho}. \tag{1.34}$$

For instance, the background metric has the form

$$\bar{g}_{\mu\nu} = \operatorname{diag}\left\{1, -a^2(t)\delta_{ij}\right\}. \tag{1.35}$$

such that in the synchronous gauge $h_{0\mu} = 0$, the (00) component of the Ricci tensor is

$$R_{00} = \bar{R}_{00} + \delta R_{00}, \tag{1.36}$$

where

$$\delta R_{00} = \frac{1}{2}\dot{h} + Hh \qquad \text{and} \qquad h = \frac{\partial}{\partial t}\left(\frac{h_{ii}}{a^2}\right). \tag{1.37}$$

On the other hand, we also would have

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}, \qquad T = \bar{T} + \delta T \tag{1.38}$$

where to compute these last perturbed quantities we will assume that entropic perturbations are negligible and that the total energy-momentum tensor is free of anisotropic stresses. Therefore, it can be expressed via

$$\bar{T}^{\mu}{}_{\nu} = (\bar{\rho}_t + \bar{p}_t)\bar{U}^{\mu}\bar{U}_{\nu} - \bar{p}_t\delta^{\mu}{}_{\nu} \tag{1.39}$$

such that

$$\bar{T}^{0}{}_{0} = \bar{\rho}_t, \qquad \bar{T}^{i}{}_{j} = -\bar{p}_t\delta^{i}{}_{j}. \tag{1.40}$$

and the 4-vector velocity U^{μ}, in the comoving coordinates is

$$\bar{U}^0 = \bar{U}_0 = 1, \qquad \bar{U}^i = \bar{U}_i = 0 \tag{1.41}$$

Therefore, considering simultaneous perturbations of the metric tensor and the total density, pressure and 4-velocity

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad U^{\alpha} = \bar{U}^{\alpha} + \delta U^{\alpha}, \tag{1.42}$$

we can find the perturbation of the energy momentum tensor as

$$\delta T_{\mu\nu} = (\bar{\rho} + \bar{p})(\delta U_\mu \bar{U}_\nu + \bar{U}_\mu \delta U_\nu) + (\delta\rho + \delta p)\bar{U}_\mu \bar{U}_\nu + \delta p \bar{g}_{\mu\nu} + \bar{p}\delta g_{\mu\nu} \tag{1.43}$$

and their mixed components in the form

$$\delta T^\mu{}_\nu = \bar{g}^{\mu\rho}\delta T_{\rho\nu} + \delta g^{\mu\rho}\bar{T}_{\rho\nu} \tag{1.44}$$

such that perturbing the alternative Einstein equations (1.11), we obtain

$$\delta R^\mu{}_\nu = 8\pi G \left(\delta T^\mu{}_\nu - \frac{1}{2} T g^{\rho\mu} \delta g_{\rho\nu} - \frac{1}{2} \delta T \delta^\mu{}_\nu \right). \tag{1.45}$$

Finally, we also need the perturbation of the conservation law (1.10)

$$\delta \left(\nabla_\mu T^\mu{}_\nu \right) = \partial_\mu \delta T^\mu{}_\nu + \Gamma^\mu{}_{\mu\rho} \delta T^\rho{}_\nu + \delta\Gamma^\mu{}_{\mu\rho} T^\rho{}_\nu - \Gamma^\rho{}_{\mu\nu} \delta T^\mu{}_\rho - \delta\Gamma^\rho{}_{\mu\nu} T^\mu{}_\rho \tag{1.46}$$

and thus we complete our set of hydrodynamical equations for the evolution of the cosmological perturbations. Inhomogeneities in the matter distribution induce metric perturbations and in the linear approximation different types of perturbations evolve independently and therefore can be analyzed separately [47]. However this set of differential equations requires suitable initial conditions to be solved with a clear physical interpretation [43].

Chapter 2

The running vacuum energy from quantum effects

In this chapter we shall review some fundamental aspects of the running vacuum energy model from the point of view of its origin in the quantum corrections to gravity in the semiclassical approach and its applications to cosmology as a dynamical form of dark energy. We shall follow here the references [15][16][48].

2.1 The semiclassical approach

This method consist in quantifying the quantum effects in a curved spacetime, formulating a quantum theory for matter fields on a classical background. Therefore, the classical actions for fields and gravity has to be well defined. Perhaps it is possible to construct a great variety of this kind of actions, we have to impose three fundamental principles: locality, general covariance and to hold the known symmetries (as gauge symmetry) in the extension to curved spacetime [16]. Additionally, we expect renormalizability and simplicity.

The most general action of vacuum, according to these principles is given by

$$S_{vac} = S_{EH} + S_{HD} \tag{2.1}$$

where the high derivatives action takes the form

$$S_{HD} = \int d^4x\sqrt{-g}\left(a_1 C^2 + a_2 E + a_3 \Box R + a_4 R^2\right) \tag{2.2}$$

with

$$C^2 = {R^2}_{\mu\nu\alpha\beta} - 2{R^2}_{\alpha\beta} + \frac{1}{3}R^2 \tag{2.3}$$

$$E = {R^2}_{\mu\nu\alpha\beta} - 4{R^2}_{\alpha\beta} + R^2 \tag{2.4}$$

as the square of the Weyl tensor and the integrand of the Gauss-Bonnet topological term, respectively. It is worthwhile to mention that these high derivatives terms are necessary to avoid problems of normalization and can be important in the primordial phase of the universe, as in the context of cosmic inflation [49][15].

2.1.1 Effective action and renormalization

To describe the quantum effects of matter fields quantization, the fundamental object is the effective action [15], which is defined by

$$e^{i\Gamma[g_{\mu\nu}]} = \int \mathcal{D}\phi \, e^{iS[\phi;g_{\mu\nu}]} \tag{2.5}$$

where ϕ is the set of all matter fields and gauge ghosts, $\mathcal{D}\phi$ is the covariant measure of the functional integration, and $S[\phi; g_{\mu\nu}]$ is the classical action, including matter fields and its possible interactions, whose form depends on the metric and the classical vacuum action (2.1).

Considering an effective action of vacuum, this is the purely metric background, we can make a loop expansion in the form [15]

$$\Gamma[g_{\mu\nu}] = S_{vac}[g_{\mu\nu}] + \bar{\Gamma}^{(1)} + \bar{\Gamma}^{(2)} + \bar{\Gamma}^{(3)} + \cdots . \tag{2.6}$$

The most important one-loop contribution is [16]

$$\bar{\Gamma}^{(1)} = \frac{i}{2} Tr \ln \hat{\mathrm{H}} \tag{2.7}$$

where

$$\hat{\mathrm{H}} = \hat{\mathrm{H}}(x,y) = \frac{1}{2}\frac{\delta^2 S[\phi, g_{\mu\nu}]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=0}. \tag{2.8}$$

The effective action is a well-defined diffeomorphism invariant quantity, such that, although we can propose alternative forms for it, we cannot include odd powers of metric derivatives, which holds for any particular metric, including the cosmological one.

On the other hand, to compute divergences and related quantities we employ the useful and successful method of Schwinger-DeWitt expansion [50], where the key point is the representation of trace

$$Tr \ln \hat{\mathrm{H}} = \ln Det \hat{\mathrm{H}} \tag{2.9}$$

using the proper time integral

$$\frac{i}{2} Tr \ln \hat{\mathrm{H}} = -\frac{i}{2} Tr \int_{\infty}^{0} \frac{ds}{s} e^{-is\hat{\mathrm{H}}} \tag{2.10}$$

where

$$e^{-is\hat{\mathrm{H}}} = \hat{U}(x,x';s) = \hat{U}_0(x,x';s)\sum_{k=0}^{\infty}(is)^k\,\hat{a}_k(x,x') \tag{2.11}$$

and $\hat{a}_k(x,x')$ are the Schwinger-DeWitt coefficients, while

$$\hat{U}_0(x,x';s) = \frac{1}{(4\pi is)^{n/2}}\mathcal{D}^{1/2}(x,x')e^{-\frac{\sigma(x,x')}{2is}-im^2s}. \tag{2.12}$$

Here $\sigma(x,x')$ is the geodesic distance between x and x' and

$$\mathcal{D}(x,x') = det\left[-\partial_\mu\partial_\nu\sigma(x,x')\right] \tag{2.13}$$

is the Van Vleck-Morett determinant [51].

The most important divergences are the logarithmic ones, which are proportional to

$$\hat{a}_2 = Tr\lim_{x'\to x}\hat{a}_2(x,x') \tag{2.14}$$

and whose form in the vacuum sector is given by [16]

$$\hat{a}_2 = \int \mathrm{d}^4x\sqrt{-g}\left(\beta_\Lambda + \beta_E R + \beta_1 C^2 + \beta_2 E + \beta_3\Box R + \beta_4 R^2\right) \tag{2.15}$$

where

$$(4\pi)^2\beta_\Lambda = \frac{1}{2}N_s m_s^4 - 2N_f m_f^4, \tag{2.16}$$

$$(4\pi)^2\beta_E = -N_s m_s^2\left(\xi-\frac{1}{6}\right) + \frac{N_f m_f^2}{3}, \tag{2.17}$$

$$(4\pi)^2\beta_4 = \frac{N_s}{2}\left(\xi-\frac{1}{6}\right)^2 \tag{2.18}$$

$$(4\pi)^2\beta_1 = \frac{1}{120}N_s + \frac{1}{20}N_f + \frac{1}{10}N_v = \omega, \tag{2.19}$$

$$(4\pi)^2\beta_2 = -\frac{1}{360}N_s - \frac{11}{360}N_f - \frac{31}{180}N_v = b, \tag{2.20}$$

$$(4\pi)^2\beta_3 = \left[\frac{1}{180} + \frac{1}{6}\left(\xi-\frac{1}{6}\right)\right]N_s + \frac{1}{30}N_f - \frac{1}{10}N_v = c, \tag{2.21}$$

and ξ is the non-minimal parameter associated to the scalar field action

$$S_{scal} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2 + \xi\varphi^2 R\right). \tag{2.22}$$

Thus, we can write the total divergent part of the effective action of vacuum theory for one-loop as [16]

$$\bar{\Gamma}^{(1)}_{div} - \frac{1}{n-4}\int \mathrm{d}^4x\sqrt{-g}\left(\beta_1 C^2 + \beta_2 E + \beta_4 R^2 + \beta_3\Box R + \beta_E R + \beta_\Lambda\right) \tag{2.23}$$

including the presence of N_s real scalars, N_f Dirac spinors and N_v massless vectors. The divergences at any loop order are of the same form as one-loop expressions, therefore at any loops we can remove the counterterms by renormalizing all the parameters of the theory, including couplings , masses, ξ and vacuum parameters [52].

In the framework of the minimal subtraction scheme (MS-scheme) and assuming dimensional regularization [15], the renormalization group equation for the effective action implies

$$\mu \frac{d\gamma}{\mathrm{d}\mu} = \left[\mu \frac{\partial}{\partial \mu} + \beta_P(n)\frac{\partial}{\partial P} + \int \mathrm{d}^n x \gamma_\Phi(n) \frac{\delta}{\delta \Phi(x)}\right] \Gamma[g_{\alpha\beta}, \Phi, P, n, \mu] = 0 \qquad (2.24)$$

where

$$\beta_P(n) = \mu \frac{\mathrm{d}P}{\mathrm{d}\mu}, \qquad \gamma_\Phi(n)\mu \frac{\mathrm{d}\Phi}{\mathrm{d}\mu} \qquad (2.25)$$

are the n-dim β and γ functions, while the usual ones correspond to the limit $n \to 4$. Here Φ are the quantized fields, $g_{\alpha\beta}$ is the classical background metric and P denotes the full set of parameters of the theory [16].

To interpret physically this renormalization group running in curved spacetime, it is necessary to understand the physical role of μ parameter. The standard procedure in the context of MS-scheme is the introduction of form factors $\sim \ln \Box/m^2$, which can be successfully applied to quantify the quantum effects in classical action of gravity in the case of, for instance, scalar field [53]. However, this is not the situation for the gravitational G and cosmological Λ constants, because the D'Alembert operator acting on them gives automatically zero. Similarly, in the case of Einstein-Hilbert term, the formfactor yields total derivatives or surface terms, which have no effect in the equations of motion [20][54][55].

Finally, from the viewpoint of GR G and Λ gain a clear physical meaning through the EFE

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \qquad (2.26)$$

so looking for their renormalization group equations

$$(4\pi)^2 \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{\lambda}{8\pi G}\right) = \beta_\Lambda - \frac{\Lambda}{2\pi G}, \qquad \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{1}{16\pi G}\right) = \beta_E - \frac{1}{8\pi G} \qquad (2.27)$$

we see that the second terms, which reflect the classical dimension of G and Λ, cancel into the EFE. Therefore one has to use only the β-functions terms in cosmological applications [16].

2.1.2 Effective action for massive fields

As we saw above, vacuum divergences can be removed by the renormalization of the action (2.1), based on the MS-scheme for the renormalization group running introducing the formfactors by trading $\ln \mu^2$ by $\ln \Box$ [56]. In the context of QED, the main phenomenon at the low-energy regime (IR) is the decoupling of massive fields [57]. We can expect a similar result in the vacuum gravitational sector.

Considering a scalar field and a resummation of the Schwinger-Dewitt series [58], it is possible to find the result for the one-loop contribution of the effective action [20][16]

$$\bar{\Gamma}^{(1)}_{scalar} = \frac{1}{2(4\pi)^2} \int d^4x \sqrt{-g} \Big\{ \frac{m^4}{2} \left(\frac{1}{\epsilon} + \frac{3}{2} \right) + \left(\xi - \frac{1}{6} \right) m^2 R \left(\frac{1}{\epsilon} + 1 \right) + \frac{1}{2} C_{\mu\nu\alpha\beta} \left[\frac{1}{60\epsilon} + k_W(a) \right] C^{\mu\nu\alpha\beta} + R \left[\frac{1}{2\epsilon} \left(\xi - \frac{1}{6} \right)^2 + k_R(a) \right] R \Big\} \tag{2.28}$$

in the $\mathcal{O}(R^2...)$ approximation. Here ϵ is the parameter of dimensional regularization

$$\frac{1}{\epsilon} = \frac{2}{4-n} + \ln \left(\frac{4\pi\mu^2}{m^2} \right) - \gamma. \tag{2.29}$$

Note that the divergences has the same form as in the MS-scheme discussed above in equation (2.15) and (2.23), but with a qualitative difference due to the presence of the non-local formfactors [16]

$$k_W(a) = \frac{8A}{15a^2} + \frac{2}{45a^2} + \frac{1}{150} \tag{2.30}$$

$$k_R(a) = A \left(\xi - \frac{1}{6} \right)^2 - \frac{A}{6} \left(\xi - \frac{1}{6} \right) + \frac{2A}{2a^2} \left(\xi - \frac{1}{6} \right) + \frac{A}{9a^2} - \frac{A}{18a^2} + \frac{A}{144} + \frac{1}{108a^2} - \frac{7}{2160} - \frac{1}{18} \left(\xi - \frac{1}{6} \right) \tag{2.31}$$

where

$$A = 1 - \frac{1}{a} \ln \left(\frac{2+a}{2-a} \right), \qquad a^2 = \frac{4\Box}{\Box - 4m^2}. \tag{2.32}$$

It can be found similar equations for the case of massive fermion and vector loops [20][59].

2.1.3 Renormalization group for the cosmological constant

Despite we get zero β-functions for the G and Λ as it can be noticed from equation (2.28), the real situation is that the corresponding $\beta_{1/G}$ and β_Λ come from the contributions of massive quantum fields, so it is natural to expect a non zero value for this functions. We can not prove that physical $\beta_{1/G}$ and β_Λ are indeed zero, therefore the conclusion has to be that also the MS-scheme and the flat-background expansion are not appropriate to quantify this running [18].

As alternative to this problem, and from the phenomenological viewpoint, we can associate the scale parameter μ with the Hubble parameter and consider applications and consequence to cosmology [48]. On the basis on covariance principle discussed before and assuming that quadratic decouplings holds for the vacuum energy, it is possible to conclude that the quantum correction for Λ take the form

$$\delta\rho_\Lambda \sim \sum_i S_i m_i^2 H^2 \qquad (2.33)$$

where the sum is over all massive particles, from the neutrino up to the possible GUT constituents and even to the hypothetical Planck-scale particles. The coefficient S_i may be different for different models and at one loop there are free fields contributions such that one can expect to have opposite signs for fermions and bosons. After this algebraic summation and adding the constant term required for renormalization, we arrive to the expression

$$\rho_\Lambda = \rho_\Lambda^0 + \frac{3\nu}{8\pi G}\left(H^2 - H_0^2\right), \qquad (2.34)$$

where ρ_Λ^0 is the present-day vacuum energy density and ν is the unique free parameter of the model. The sign and magnitude of ν depend on the mass spectrum of the particle physics model, according to (2.32).

Although we have not definitive arguments to assert that quadratic decoupling takes place, what it is possible to claim is that, in the absence of cosmic scalar field (or another form of dark energy), the equation (6.60) is the unique possible form for a non constant vacuum energy [16].

Two clarifying observations considering the definition of our model, are in order at this point. First of all, due to the Planck suppression, the fourth- and higher-derivative terms in the classical action and loop corrections are irrelevant even at the relatively high energy scale, such as the one we deal with in this paper. To understand this, let us quote the Starobinsky model of inflation [49][60]. This model is mostly based on the higher derivative R^2-addition to the Einstein-Hilbert action. In the presence of anomaly-induced terms, there is a solution with a constant Hubble parameter H [49]. However, in the course of inflation H decreases (approximately linearly with time) and its magnitude at the end of the inflationary epoch is supposed to be about $10^{13}\,GeV$. This provides a sufficient difference between the effect of the intensive running of the cosmological constant which we shall explore and the effect of the R^2-term. Indeed, the running Λ in the presence of the R^2-term may be relevant in earlier phases of inflation, but this is another issue to study and we leave it to the future work.

Assuming the simultaneous energy exchange between the cosmological constant density and Einstein-Hilbert sector and between the cosmological constant density and matter sector leads to an ambiguity in the cosmological model. Due to this feature, the models with double energy exchange were never elaborated, regardless on an extensive literature on the running cosmological models (see e.g. [10]). Besides the mentioned ambiguity, from the practical side there is no much sense in considering such a double energy exchange, because the effect of the running of G is known to be much weaker than the one of the energy exchange with matter [24].

The main problem with the model based on the vacuum-matter energy exchange is that during most of the history of the Universe the typical energies of the gravitational degrees of freedom are very small compared to the masses of all known particles [61]. For instance, the value of the Hubble parameter today is about $H_0 \propto 10^{-42}$ GeV, while the lightest neutrino is supposed to have the mass about thirty orders of magnitude greater. Thus, there is only a possibility to create photons and this is not phenomenologically interesting, since the energy density of such photons would be about T^4, with the temperature $T \approx H$. Such an energy density is of course much smaller than the energy density of CMB, which is yet about four orders of magnitude smaller compared to the present-day critical density, or to the cosmological constant density. This argument represents a serious obstacle to using this model for a *late* cosmology.

Let us note that the described restrictions do not apply to the *early* Universe, e.g., to the epoch after inflation, where the value of the Hubble parameter is decreasing from about 10^{13} GeV to the values that are comparable to the energy scale of the Minimal Standard Model of elementary particle physics. This is a reheating period, where the creation of particles is very intensive, and there is nothing wrong with assuming that this happens because of the decay of the cosmological constant into the matter. In the next section, we shall explore the model [21] in the high energy domain. The description of quantum effects is based on the running of cosmological constant described in this paper. At the same time, the models of early Universe require special care about the description of matter. The matter contents of the Universe consist mainly of the usual matter particles and DM. We assume that the DM consists of the GUT remnants and hence has masses that are much greater than the value of H. Thus, the DM can be regarded to decouple, in the sense that DM particles are not created from the vacuum. Thus, an appropriate description of DM is an ideal gas of massive particles adiabatically expanding and becoming less relativistic with time. To describe such a gas, we shall use the simple and convenient Reduced Relativistic Gas (RRG) model, which was originally developed by Sakharov in the classical paper [30], and recently reinvented in [2][3].

2.2 Cosmology with a running vacuum energy

Let us review now a simple cosmological model including the running of the CC, following the reference [21]. Considering also the presence of usual non-relativistic matter the Friedmann equation takes the form

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_\Lambda\right) - \frac{k}{a^2} \tag{2.35}$$

and expressing the running of the CC in terms of its beta function

$$\frac{d\rho_\Lambda}{dH} = \frac{1}{H}{\beta^t}_\Lambda \tag{2.36}$$

we can write

$$\frac{d\rho_\Lambda}{dz} = \frac{1}{H}\frac{dH}{dz}{\beta^t}_\Lambda = \frac{3\nu}{4\pi G}H\frac{dH}{dz} \tag{2.37}$$

where it has been defined the dimensionless running parameter

$$\nu = \frac{\sigma M^2}{12\pi M_p^2}. \tag{2.38}$$

Additionally, we have a modified conservation law as given by the interaction between matter sector and this running vacuum, in the form

$$\frac{d\rho_m}{dz} + \frac{d\rho_\Lambda}{dz} - \frac{3(\rho_m + p_m)}{1+z} = 0 \tag{2.39}$$

Therefore, the system of equations for this model can be cast as [21]

$$H^2(z) \;=\; \frac{\kappa^2}{3}\big[\rho_\Lambda(z) + \rho(z)\big] + H_0^2\Omega_k^0(1+z)^2, \tag{2.40}$$

$$\frac{d\rho_m}{dz} - \frac{3(1+w)}{1+z}\,\rho = -\frac{d\rho_\Lambda}{dz}, \tag{2.41}$$

$$\frac{d\rho_\Lambda}{dz} \;=\; \frac{3\nu}{8\pi G}\frac{dH^2}{dz}, \quad . \tag{2.42}$$

where we have used

$$p_m = w\rho_m, \qquad -\frac{k}{a^2} = H_0^2{\Omega^0}_k(1+z)^2. \tag{2.43}$$

The solution of this system can be obtained analytically by following the procedure described in [62], such that we can find closed expression for $H(z)$, ρ_m and ρ_Λ. Let us summarize the main steps as follows. First, to obtain $\rho_m(z)$ one has to consider the derivative of Friedmann equation and then use the equation for the ρ_Λ running such that we get

$$\frac{d\rho_\Lambda}{dz} = \frac{3\nu(1+w)\rho_m}{(1+z)} + \frac{3\nu H_0^2}{4\pi G}{\Omega^0}_k(1+z). \tag{2.44}$$

Using now the conservation law, one finally obtains the differential equation

$$\frac{d\rho_m}{dz} - \frac{\zeta\rho_m}{(1+z)} = \gamma\,{\rho^0}_c\,(1+z) \tag{2.45}$$

where we have defined the constants

$$\zeta \;=\; 3(1+w)(1-\nu), \qquad \gamma = -2\nu{\Omega^0}_k \tag{2.46}$$

and used the definition of the critical density today as

$${\rho^0}_c = \frac{3H_0^2}{8\pi G}. \tag{2.47}$$

The solution of this differential equation (2.45) is given by

$$\rho_m(z) = \left(\rho^0{}_m + \frac{\gamma\rho^0{}_c}{\zeta - 2}\right)(1+z)^\zeta - \frac{\gamma}{\zeta-2}\rho^0{}_c(1+z)^2, \tag{2.48}$$

and with this solution we can find $\rho_\Lambda(z)$ using

$$\frac{d\rho_\Lambda}{dz} = \frac{3(1+w)}{(1+z)}\rho - \frac{d\rho_m}{dz} \tag{2.49}$$

which after integration yields

$$\rho_\Lambda(z) = \rho^0{}_\Lambda + \rho^0{}_m f(z) + \rho^0{}_c g(z) \tag{2.50}$$

with

$$f(z) = \frac{1}{1-\nu}\left[(1+z)^\zeta - 1\right] \tag{2.51}$$

and

$$g(z) = -\frac{\gamma(1+3w)}{2(\zeta-2)}z(z+2) + \nu\gamma\frac{(1+z)^\zeta - 1}{(1-\nu)(\zeta-2)}. \tag{2.52}$$

Finally, using both of the solutions, it is possible to show that the Hubble parameter takes the form

$$H^2(z) = H_0^2\left\{1 + \left(\Omega^0{}_m - \frac{2\nu\Omega^0{}_k}{\zeta-2}\right)\left[\frac{(1+z)^\zeta - 1}{1-\nu}\right] + \frac{3\nu\Omega^0{}_k(1+w)z(z+2)}{\zeta-2}\right\} \tag{2.53}$$

thus completing the solution of our system of cosmological equations including the running of the CC. Notice that the only free parameter of the model is the running parameter ν, which should be constrained using some cosmic observables [62][25][63].

However, as a first step, we can estimate a constraint for ν from theoretical arguments. In order to satisfy the Big bang nucleosyntheis (BBN) requirements, where the universe is dominated by the radiation component (which would be equivalent to take $w = 1/3$ in our model), we have that, in this RD regime ($T \gg T_0$), the solutions can be written as

$$\rho_r(T) = \frac{\pi^2 g_*}{30}\left(r^{-\nu}T\right)^4 - \frac{\nu}{1-2\nu}\left[r^{4(1-\nu)} - r^2\right]\Omega_k^0\rho_c^0 \tag{2.54}$$

and

$$\rho_\Lambda(T) \approx \frac{\nu}{1-\nu}\frac{\pi^2 g_*}{30}\left(r^{-\nu}T\right)^4 - \frac{\nu}{1-2\nu}\left[\frac{\nu}{1-\nu}r^{4(1-\nu)} - r^2\right]\Omega_k^0\rho_c^0 \tag{2.55}$$

Such that, according to BBN, one needs to guarantee that

$$|\rho_\Lambda/\rho_r| \approx |\frac{\nu}{1-\nu}| \ll 1 \tag{2.56}$$

which immediately implies that

$$|\nu| \ll 1 \tag{2.57}$$

so the running parameter ν has to be small. To have more conclusive and quantitative results about this qualitative constraint, is necessary to consider some cosmic datasets (SNIa, BAO or the firs acoustic peak of the CMB) or the behavior of the running model in the presence of perturbations and compare with some observables like the matter power spectrum [1][63][24] [34][64]. Let us make now some comments about the inclusion of cosmological perturbations in this running model.

2.3 Density perturbations for the running

It was already mentioned that considering the presence of cosmic perturbations implies perturbations in the metric as well as in matter sector (see Sect. 1.2). Therefore, in first place, we need the 00 component of the Einstein equations, which for this case is

$$\dot{h} + 2Hh = 8\pi G\left[(1+3w)\delta\rho_m - 2\delta\rho_\Lambda\right] \tag{2.58}$$

To compute the perturbation of the running density $\delta\rho_\Lambda$ in a covariant form, remember that we can express it as

$$\rho_\Lambda = A + B(3H)^2 \tag{2.59}$$

where

$$A = \rho_\Lambda^0 - \frac{3\nu}{8\pi}M_p^2 H_0^2, \qquad B = \frac{\nu M_p^2}{24\pi} \tag{2.60}$$

but noting that with a very simple calculation, we can show the relation

$$\nabla_\mu \bar{U}^\mu = \partial_\mu \bar{U}^\mu + \Gamma^\mu{}_{\mu\nu}\bar{U}^\nu = 3H \tag{2.61}$$

such that

$$\delta\left(\nabla_\mu \bar{U}^\mu\right) = 3\delta H = \nabla_i(\delta\bar{U}^i) + \delta\Gamma^i{}_{i0} = \theta - \frac{h}{2} \tag{2.62}$$

with the definitions

$$\theta = \nabla_i(\delta\bar{U}^i), \qquad h = \frac{\partial}{\partial t}\left(\frac{h^i{}_i}{a^2}\right). \tag{2.63}$$

Therefore we can rewrite the running density as

$$\rho_\Lambda = A + B(\nabla_\mu U^\mu)^2 \tag{2.64}$$

and its perturbation

$$\delta\rho_\Lambda = B\delta\left[(\nabla_\mu \bar{U}^\mu)^2\right] = 2B(\nabla_\mu \bar{U}^\mu)\delta(\nabla_\mu \bar{U}^\mu) \tag{2.65}$$

but using (2.61) and (2.62), we finally get

$$\delta\rho_\Lambda = 6BH\left(\theta - \frac{h}{2}\right). \tag{2.66}$$

To complete our perturbative description of the running model, we take the $\nu = 0$ and $\nu = i$ components of the perturbed conservation equation $\delta(\nabla_\mu T^{\mu\nu})$, which yield

$$\delta\dot{\rho}_m + \left(\theta - \frac{h}{2}\right)\rho_m + 3H\delta\rho_m = -\delta\rho_\Lambda \tag{2.67}$$

$$\dot{\rho}_m\theta + 5H\rho_m\theta + \rho_m\dot{\theta} = -\frac{k^2}{a^2}\delta\rho_\Lambda \tag{2.68}$$

where we have taken the Fourier transform in the second equation and k is the wave number and it makes no reference to spatial curvature here. The equations (2.58), (2.67), together the constraint (2.66), form a complete and solvable system of equations, where we have to use the expression for the Hubble parameter found in previous section (2.53). To solve this system numerically, it is useful to write them in terms of another variables

$$v = f_1\nabla_i(\delta\bar{U}^i) \qquad f_1 = \frac{\rho_m}{\rho_t}, \qquad f_1 = \frac{\rho_\Lambda}{\rho_t} \tag{2.69}$$

such that

$$\delta_\Lambda = \frac{g}{f_2}\left(\frac{v}{f_1} - \frac{h}{2}\right). \tag{2.70}$$

and our system now, in terms of the density contrast δ_m and the redshift z takes the form

$$v' + \left[\frac{(3f_1 - s)}{(1+z)} - \frac{k^2\varrho(1+z)}{Hf_1}\right]v = \frac{k^2\varrho(1+z)}{2H}h \tag{2.71}$$

$$h' + \frac{2(\nu-1)}{(1+z)}h = \frac{2\nu}{(1+z)}\left(\frac{2v}{f_1} - \frac{f_1}{\varrho}\delta_m\right) \tag{2.72}$$

$$\delta_m' + \left(\frac{f-1'}{f_1} - \frac{3f_2}{(1+z)}\right)\delta_m = \frac{1}{f_1}\left(\frac{\varrho h}{2} - \frac{\varrho v}{f_1}\right)' + \frac{1}{(1+z)}\left(3\varrho - \frac{1}{H}\right)\left(\frac{v}{f_1} - \frac{h}{2}\right) \tag{2.73}$$

where

$$\delta_m = \frac{\delta\rho_m}{\rho_m}, \qquad \varrho = \frac{2\nu H(z)}{3H^2(z) - 3H_0^2{\Omega^0}_k(1+z)^2}. \tag{2.74}$$

Thus, we a complete set of coupled Fourier modes for the velocity, density contrast and metric perturbations for running CC model. In the case of $\nu = 0$ they reproduce the situation in the ΛCDM model [1]. From the numerical solution of the system it is possible to reconstruct the matter power spectrum for arbitrary ν and compare with observational data, finding new constraints for this running parameter. In figure 2.1 we show the plot of the matter power spectrum for different values of ν, where it is possible to see that the larger is ν the more suppressed is the power, mainly at large scales (large k). As it was estimated in last section ν is small and must have values in the interval $10^{-8} < \nu < 10^{-6}$ in order to fit the observational data. The larger values are practically ruled out by this test.

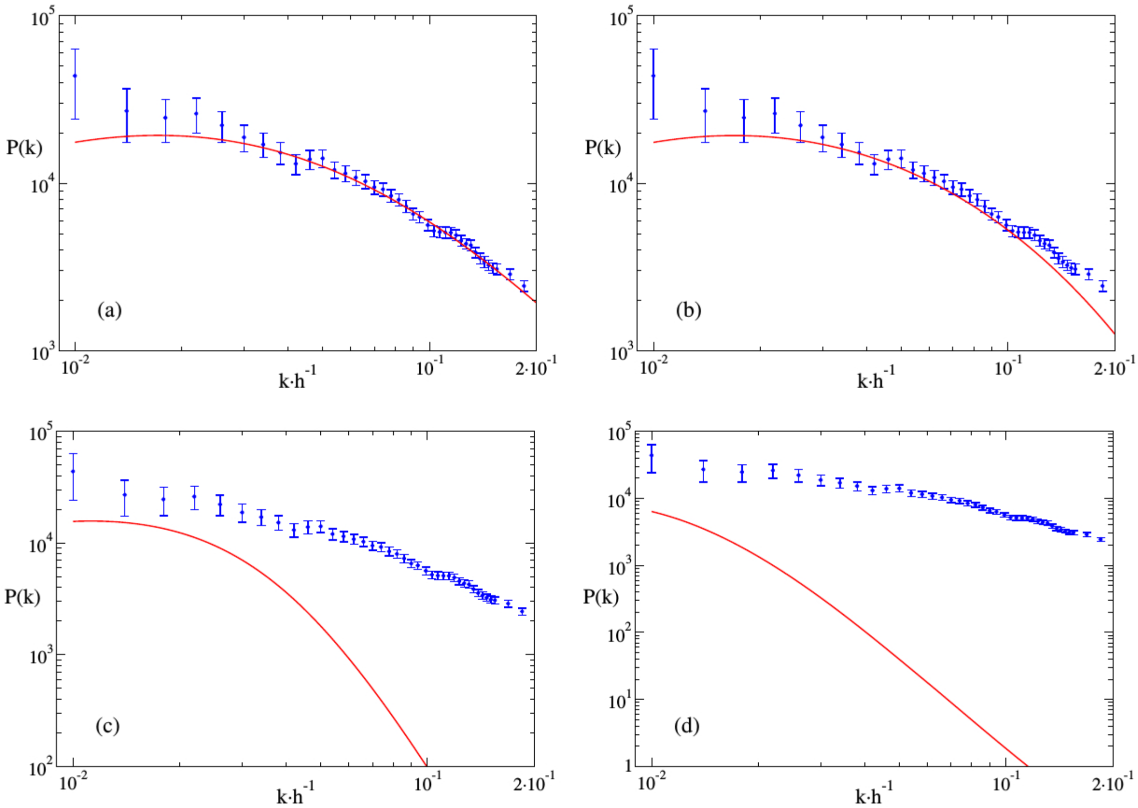

Figure 2.1: Power spectrum for the RRG-Λ model for fixed $\Omega_B^0 = 0.04$, $\Omega_{DM}^0 = 0.21$ and $\Omega_\Lambda^0 = 0.75$, with the values $b = 10^{-5}$, $b = 10^{-4}$, $b = 2 \times 10^{-4}$ and $b = 10^{-3}$. The theoretical plots are presented together the LSS data from the 2dFGRS. Figure taken from [1]

From this example of numerical analysis of the running model and its validity in the light of the matter power spectrum, we can estimate the effect of this running on matter perturbations evolution in different epochs and explore the possibility of having new cosmological results beyond the standard ΛCDM.

Chapter 3

Warm dark matter as a reduced relativistic gas

In this chapter we present a warm dark matter (WDM) description using the simple and useful model of the reduced relativistic gas (RRG), discussing its main properties and some cosmological applications at background as well as perturbative levels. We shall follow here the references [3][2][65].

3.1 The reduced relativistic gas equation

The equation of state for the RRG is one of several examples of fluids whose state parameter is not a constant, but it has some dependence with time [40], the field itself [66] or even the scale factor [67]. In order to show the two well known forms of the RRG equation of state, we follows and summarize here the procedure of the references [2][65].

Let us consider first a set of N relativistic particles of mass m in a box of volume V. The time average of the pressure produced by the particles on the walls of the box is given by

$$p = \frac{n\epsilon}{3V}\frac{v^2}{c^2} \tag{3.1}$$

with

$$\epsilon = \frac{m\, c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad n = \frac{N}{V}, \tag{3.2}$$

and we have the relativistic dispersion relation

$$\epsilon^2 = P^2c^2 + m^2c^4 \tag{3.3}$$

where P is the momentum of the particle. On the other hand, the energy density would be

$$\rho = n\epsilon = \frac{n\, m\, c^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{3.4}$$

and therefore we can write the first form for the equation of state for the RRG

$$p = \frac{\rho}{3}\left[1 - \left(\frac{mc^2}{\epsilon}\right)^2\right]. \tag{3.5}$$

Introducing now the notations

$$\rho_1 = Nmc^2/V_0, \qquad \rho_d = \rho_1 V_0/V = Nmc^2/V, \tag{3.6}$$

where V_0 is some fixed initial value of the volume, we can rewrite the equation of state in its second form as

$$p = \frac{\rho}{3}\left(1 - \frac{\rho_d^2}{\rho^2}\right) \tag{3.7}$$

where it is very clear now that $w = p/\rho = w(\rho)$, so is not constant.

It is also possible to derive the RRG equation, remembering that, in phase space, the number of particles can be written in terms of the distribution function as

$$N = \int d^3x\, d^3p\, f(x,p) \tag{3.8}$$

Now, instead of considering a Maxwell-Boltzmann distribution, it is supposed equal kinetic energy ϵ_0 for all particles [65], such that

$$f(x,p) = f_0\delta(\epsilon - \epsilon_0), \tag{3.9}$$

where f_0 is a constant. Coming back to the expression for N, we have

$$N = f_0 \int d^3x\, d\Omega\, p^2\, dp\delta(\epsilon - \epsilon_0) \tag{3.10}$$

but

$$\epsilon^2 = p^2c^2 + m^2c^4, \qquad c^2p^2dp = \sqrt{\epsilon^2 - m^2c^4}E\, dE \tag{3.11}$$

so

$$N = \frac{f_0}{c^2}\int d^3x d\Omega\, \sqrt{\epsilon^2 - m^2c^4}\epsilon\, d\epsilon\, \delta(\epsilon - \epsilon_0) \tag{3.12}$$

which after integration, it is possible to find

$$f_0 = \frac{nc^2}{4\pi\epsilon_0\sqrt{\epsilon_0^2 - m^2c^4}}, \tag{3.13}$$

and thus the distribution function takes the form

$$f(E) = \frac{nc^2}{4\pi\epsilon_0\sqrt{\epsilon_0^2 - m^2c^4}}\delta(\epsilon - \epsilon_0). \tag{3.14}$$

The energy-momentum tensor can be written as

$$T^{\mu}_{\ \nu}(x,p) = \int d^3p \, \frac{p^{\mu} \, p_{\nu}}{\epsilon} f(x,p) \tag{3.15}$$

such that

$$T^{0}_{\ 0} = \rho(\epsilon) = \int d^3 \, p \, \epsilon f(\epsilon) \tag{3.16}$$

$$= \int d\epsilon \, d\Omega \epsilon^2 \sqrt{\epsilon^2 - m^2 c^4} f_0 \delta(\epsilon - \epsilon_0) = nc^2 \, \epsilon_0 \tag{3.17}$$

or

$$\rho(E) = nc^2 \epsilon_0 \tag{3.18}$$

The spatial component would be

$$T^{i}_{\ j} = \int d^3 p p^2 \hat{p}^i \hat{p}_j \frac{f(\epsilon)}{\epsilon} \tag{3.19}$$

and taking the trace

$$T^{i}_{\ i} = 3p(\epsilon) = \int d^3 p \, p^2 \frac{f(\epsilon)}{\epsilon} = \int d\epsilon \, d\Omega \, f_0 \left(\epsilon^2 - m^2 c^4\right)^{3/2} \delta(\epsilon - \epsilon_0). \tag{3.20}$$

By integrating we get the result

$$p(\epsilon) = \frac{n}{3\epsilon_0} \left(\epsilon_0^2 - m^2 c^4\right), \tag{3.21}$$

and defining $\rho_d = nmc^2$ and using the expression for the energy density we see that

$$p = \frac{\rho}{3} \left(1 - \frac{\rho_d^2}{\rho^2}\right) \tag{3.22}$$

which is exactly the equation of state for the RRG obtained previously using more simple arguments (3.7).

3.2 Comparing with Jüttner model

Let us consider now the relativistic correction of the Maxwell-Boltzmann distribution for this ideal gas of massive particles [68]. The partition function for a single particle is given by the expression

$$Z = \int d^3 \, pd^3 \, qe^{-\epsilon/kT} = 4\pi m^2 cV \, K_2 \left(\frac{mc^2}{kT}\right) \tag{3.23}$$

where $K_l(x)$ is a modified Bessel function of index l. The equation of state of N particles can be derived as usual in the form

$$p_J V = kTN \left(\frac{\partial \ln Z}{\partial \ln V}\right) \Big|_T = NkT \tag{3.24}$$

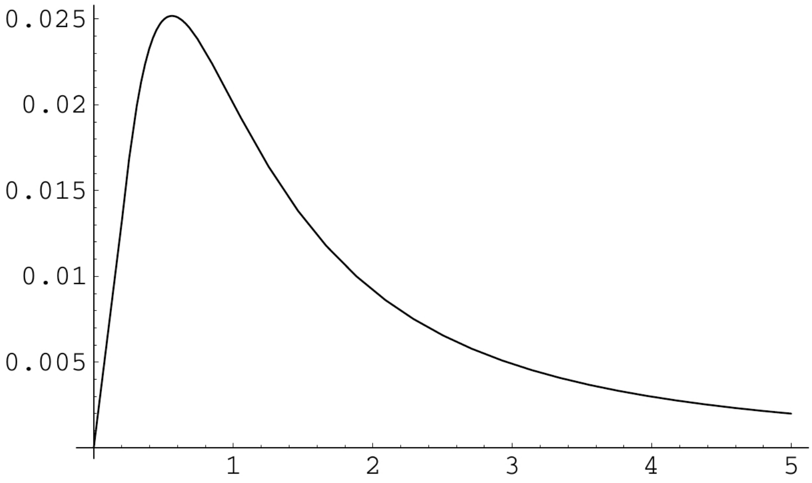

Figure 3.1: Difference between RRG and original Jüttner models. The maximum discrepancy is about 2, 5%. Figure taken from [2].

while the average energy of the particle is

$$\bar{\epsilon} = \frac{1}{Z}\int d^3p\, d^3q\, e^{-\epsilon/kT}\epsilon = mc^2 \frac{K_3\left(\frac{mc^2}{kT}\right)}{K_2\left(\frac{mc^2}{kT}\right)} - kT. \tag{3.25}$$

Now, to see the difference between the RRG and the original relativistic Jüttner description, let us solve the equation (3.24) for kT and substitute the result in (3.25) to obtain

$$\rho_J = \rho_d \frac{K_3\left(\frac{mc^2}{kT}\right)}{K_2\left(\frac{mc^2}{kT}\right)} - p_J. \tag{3.26}$$

We can also rewrite the RRG equation of state (3.7) as

$$\rho_R = \frac{3}{2}\rho_R + \sqrt{\rho_d^2 + \frac{9}{4}\rho_R^2} \tag{3.27}$$

such that if we define the dimensionless quantity [2]

$$\delta_\rho = \frac{|\rho_R - \rho_J|}{\rho_J} \tag{3.28}$$

it is possible to study numerically and to find a maximal difference of approximately 2, 5% as is showed in the figure 3.1.

3.3 Simple cosmological model using the RRG

As in the case of perfect fluid, whose equation of state $p = w\rho$, where w is a constant, we are also interested in determining the cosmological evolution of the

RRG energy density using the correspondent conservation law. Then, substituting the expression for the pressure (3.7) in (1.15) (in terms of the redshift z)

$$\frac{d\rho}{dz} - \frac{3}{(1+z)}(\rho + p) = 0 \tag{3.29}$$

we get a Bernoulli equation in the form

$$\frac{d\rho}{dz} - \frac{4}{(1+z)}\rho = -\rho_1(1+z)^5\rho^{-1} \tag{3.30}$$

whose solution is

$$\rho(z) = \sqrt{\rho_1^2(1+z)^6 + \rho_2(1+z)^8} \tag{3.31}$$

or in terms of scale factor

$$\rho(a) = \sqrt{\rho_1^2\left(\frac{a_0}{a}\right)^6 + \rho_2\left(\frac{a_0}{a}\right)^8} \tag{3.32}$$

where we have used the initial condition $\rho(a_0) = \sqrt{\rho_1^2 + \rho_2^2}$, ρ_1 is the rest energy at some initial point $V = V_0$ and ρ_2 can be interpreted as the radiation component of the RRG [2]. This solution can be also cast in the form

$$\rho_R(a) = \frac{\rho_1}{a^3}\sqrt{1 + \frac{b^2}{a^2}} \tag{3.33}$$

where

$$b^2 = \frac{\rho_2^2}{\rho_1^2} \tag{3.34}$$

is the warmness parameter b, which can also be expressed as

$$b = \frac{v/c}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.35}$$

such that, for a low warmness $v/c \ll 1$, we have $b \sim v/c$. Thus, $b \approx 0$ means that the matter content is "cold". The non-relativistic (NR) and ultra-relativistic (UR) limits correspond to the cases when $\rho_2 = 0$ and $\rho_1 = 0$, respectively, setting up the well known interpolation between radiation and dust components for the RRG [3]. We can also write these limits in the form

$$UR: \begin{cases} \rho_1 \to 0 \\ a \to 0 \end{cases} \Rightarrow \frac{a^2}{b^2} \ll 1, \tag{3.36}$$

$$NR: \begin{cases} \rho_2 \to 0 \\ a \to \infty \end{cases} \Rightarrow \frac{b^2}{a^2} \ll 1 \tag{3.37}$$

such that, the equation of state parameter $\omega_R(\rho) = p_R/\rho_R$ for this RRG satisfies

$$\lim_{a\to\infty} \omega_R(\rho) = \omega_{nr}(\rho) = 0, \qquad \lim_{a\to 0} \omega_R(\rho) = \omega_{ur}(\rho) = \frac{1}{3} \tag{3.38}$$

representing exactly the aforementioned interpolation. Let us review briefly in the following subsections, the main cosmological properties of this RRG and its simple variants.

3.3.1 Pure RRG

For this case, we can find the behavior of the scale factor and compare with the NR and UR limits of the model. From Friedmann equation (1.14) we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{3.39}$$

with the solution

$$\left(a^2 + \frac{\rho_2^2}{\rho_1^2}\right)^{3/4} = \sqrt{6\pi G}t \tag{3.40}$$

where is very clear that in the NR limit, when $\rho_2 \to 0$, we obtain

$$a \sim t^{2/3}, \tag{3.41}$$

as expected for a non-relativistic matter. The UR case is singular, however, we can consider that $\rho_1 \ll \rho_2$ and expand in series the ratio ρ_1/ρ_2, obtaining

$$a(t) = \tilde{a}\sqrt{t - t_0} \tag{3.42}$$

with

$$\tilde{a} = \sqrt{\frac{32\pi G\rho_2}{3}}, \qquad t_0 = \frac{\rho_2^{3/2}}{\sqrt{6\pi G}\rho_1^2} \tag{3.43}$$

so we have

$$a(t) \sim t^{1/2} \tag{3.44}$$

as expected, again, for a radiation component.

3.3.2 RRG and radiation

In this case the Friedmann equation takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho + \rho_\gamma) \tag{3.45}$$

whose solution can be cast in the form

$$a^2 - \frac{1}{3}\left(\frac{\rho_1}{\rho_{\gamma 0}}\right)\left(a^2 + b^2\right)^{3/2} = \sqrt{\frac{32\pi G\rho_{\gamma 0}}{3}}t. \tag{3.46}$$

Considering that $\rho \ll \rho_\gamma$, which is the case for a radiation dominated (RD) universe, we can interpret the effect of the RRG as a small correction of the $a \sim t^{1/2}$ law for a radiation component. In the figure 3.2 we show the plot of the scale factor for this model in contrast with the standard radiation and dust-like matter cases.

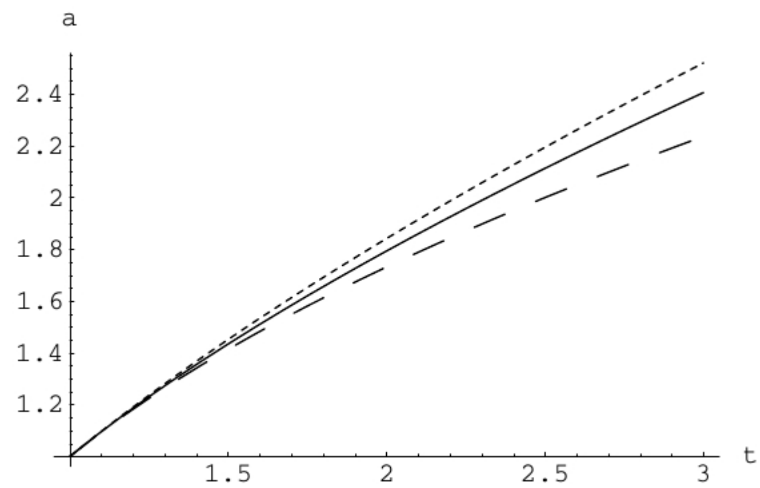

Figure 3.2: Scale factor for the cases: pure radiation ($a(t) \sim t^{1/2}$), pure dust-like ($a(t) \sim t^{2/3}$) and RRG+Radiation, corresponding to dashed, dotted and continuous lines, respectively. Figure taken from [2]

3.3.3 RRG and cosmological constant

Here, the correspondent Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) \tag{3.47}$$

whose approximate solution can be written as

$$\ln a + X(a) + \lambda t \tag{3.48}$$

with the $X(a)$ function as

$$X(a) = \frac{\rho_1}{8\rho_\Lambda}\left[\frac{\sqrt{a^2+b^2}}{a^4} + \frac{\sqrt{a^2+b^2}}{2b^2a^4} + \frac{1}{4b^3}\ln\left(\frac{\sqrt{a^2+b^2}-b}{\sqrt{a^2+b^2}+b}\right)\right] \tag{3.49}$$

In the figure 3.3 we show the plot of the scale factor for this model in contrast with the standard CC+dust-like matter and CC+radiation cases.

3.4 Including perturbations

It is also possible to consider the evolution of the RRG-Λ model in the presence of perturbations following the approach developed in Refs. [1] and [3]. This implies simultaneous perturbations of metric, energy density and the four-velocitiy in the co-moving coordinates, such that

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}, \quad \rho_R \to \rho_R + \delta\rho_R, \quad U^\alpha \to U^\alpha + \delta U^\alpha, \tag{3.50}$$

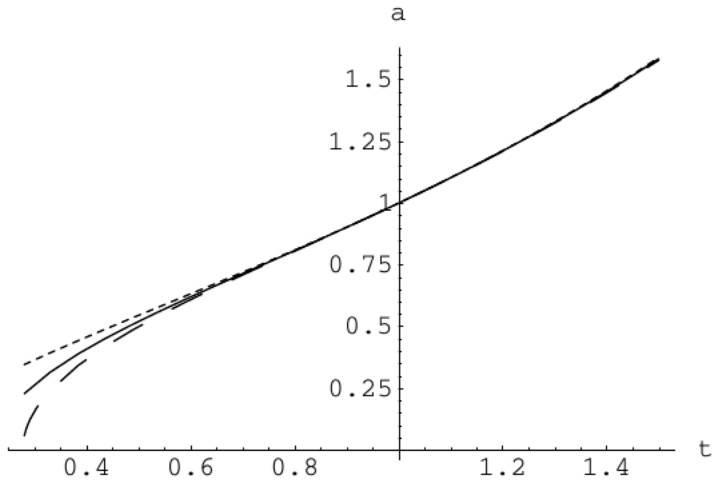

Figure 3.3: Scale factor for the cases: CC+dust-like matter, CC+radiation and CC+RRG, corresponding to dotted, dashed and continuous line, respectively. Figure taken from [2]

where U^α is the WDM velocity. The perturbation of the WDM pressure should be derived from the equation of state (3.5),

$$\delta P_{dm} = \frac{\delta\rho_{dm}}{3}\left[1 - \left(\frac{mc^2}{\varepsilon}\right)^2\right] = \frac{\delta\rho_{dm}(1-r)}{3}, \tag{3.51}$$

meaning that the perturbations satisfy the same equation of state as the background quantities. Technically, this means that the variations of the energy density $\delta\rho_{dm}$ and the rest energy density $\delta\rho_d$ are always proportional. The reason for this restriction is that in the framework of the RRG model one has to provide kinetic energies of all particles to be equal and, therefore, we have no right to change the ratio mc^2/ϵ [3]. In the synchronous gauge $h_{0\mu} = 0$ and using the constraint $\delta U^0 = \delta V^0 = 0$, the system of equations for the perturbed quantities is given by

$$\delta p = \frac{\delta\rho}{3}(1-s) \tag{3.52}$$

$$h' - \frac{2h}{(1+z)} = -\frac{f_1(2-s)}{g}\delta \tag{3.53}$$

$$\delta' - \frac{1}{(1+z)}\left[4 - s - \frac{(1+z)\rho'}{\rho}\right]\delta + \frac{4-s}{3H(1+z)}\left(\frac{h}{2} - \frac{v}{f_1}\right) = 0 \tag{3.54}$$

$$v' + \left(\frac{\rho'}{\rho} - \frac{s'}{4-s} - \frac{5}{1+z} - \frac{f_1'}{f_1}\right)v + \frac{k^2(1+z)f_1}{H}\frac{1-s}{4-s}\delta = 0 \tag{3.55}$$

where

$$v = f_1(\nabla_i \delta U^i), \qquad f_1 = \frac{\rho}{\rho_t} \tag{3.56}$$

and

$$\delta(\mathbf{x}, z) = \int \frac{d^3k}{(2\pi)^3}\delta(\mathbf{k}, z)e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad v(\mathbf{x}, z) = \int \frac{d^3k}{(2\pi)^3}v(\mathbf{k}, z)e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad k = |\mathbf{k}| \tag{3.57}$$

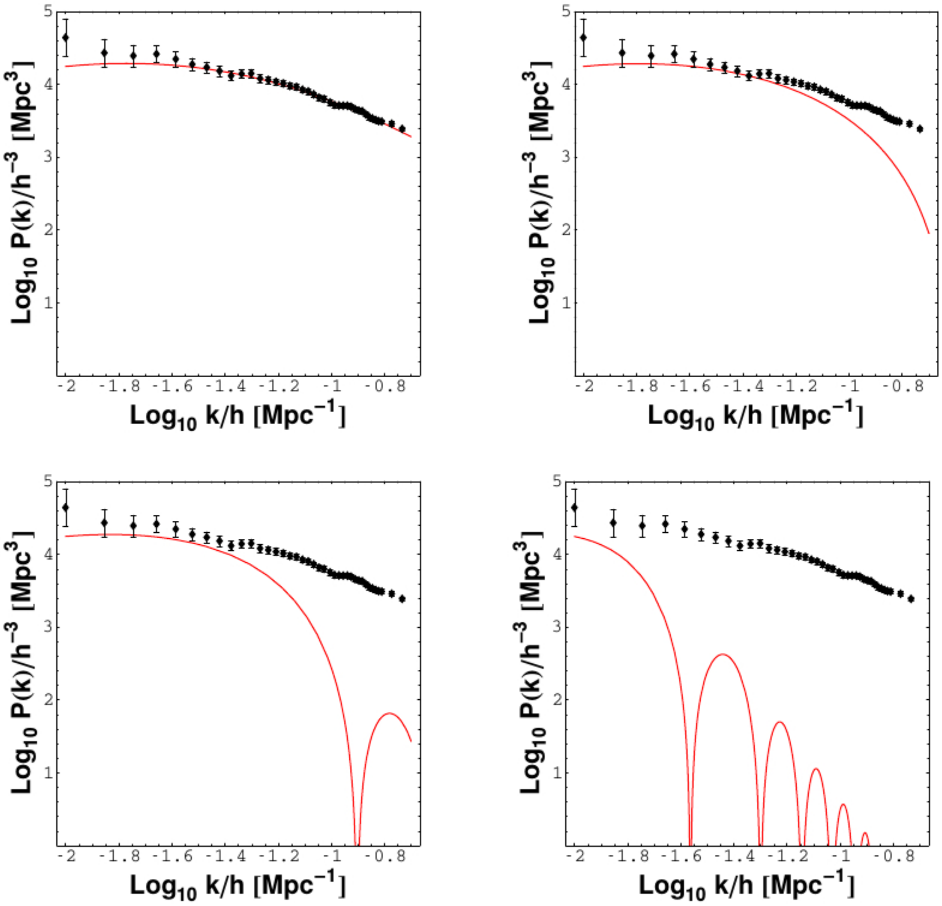

Figure 3.4: Power spectrum for the RRG-Λ model for fixed $\Omega_B^0 = 0.04$, $\Omega_{DM}^0 = 0.21$ and $\Omega_\Lambda^0 = 0.75$, with the values $b = 10^{-5}$, $b = 10^{-4}$, $b = 2 \times 10^{-4}$ and $b = 10^{-3}$. The theoretical plots are presented together the LSS data from the 2dFGRS. Figure taken from [3]

It is possible to solve this system of equations numerically and to reconstruct the matter power spectrum, defined by

$$\mathcal{P} = \delta_k^2 \tag{3.58}$$

in order to compare with observational data, for instance, using the 2dFGRS [69]. The quantity δ_k is the component of the Fourier transform of the density contrast $\delta(t)$, computed by integrating the system of equations for a given value of k and with a given initial conditions [1]. In the figure 3.4 we show the plots of the reconstructed matter power spectrum for the above system, for several values of the warmness paramater b and in contrast to the 2dFGRS data. From the plots is possible to see that the small constraint $b \sim 10^{-5}$ fits quite well the observational data while bigger values for b can be practically ruled out.

Chapter 4

Running vacuum energy and warm dark matter

In this chapter we present a cosmological model where is considered the presence of a running vacuum energy and a warm dark matter (WDM) component in the form of a reduced relativistic gas (RRG). We study the behavior at background and perturbative levels, comparing and testing its predictions against some well known cosmological observables as the CMB first acoustic peak and the matter power spectrum.

4.1 Background solution

We consider a cosmological model with the possibility of particle creation in the primordial Universe due to the quantum effects of vacuum. More precisely, we study the vacuum energy decay as a result of the renormalization group (RG) equation for the density of the cosmological constant term as it was discussed in 2.

In what follows primes indicate derivatives with respect to the redshift parameter

$$1 + z = \frac{a_0}{a}. \tag{4.1}$$

The matter contents of the Universe include usual matter, DM and radiation, according to the current estimate [46]. Here, the WDM component is described as a reduced relativistic gas (RRG) of massive particles, which take into account in a simple and useful way the warmness of the fluid and for the usual matter we assume an ultra-relativistic equation of state $P_b \approx \frac{1}{3}\rho_b$. Here we consider an early post-inflationary Universe, where the WDM has already decoupled from the other matter components and satisfies a proper continuity equation in the form

$$\rho'_w - \frac{(4-s)}{1+z}\,\rho_w = 0. \tag{4.2}$$

In the early Universe, one can restrict the consideration by the spatially flat FLRW metric ($k = 0$). The solution for Eq. (4.2) can be easily found for a single adiabatically expanding fluid [3]. Then the scaling law for the relative energy density (relative to the critical density today[1]) for the relativistic gas representing the WDM, is given by the expression

$$\Omega_w(z) \,=\, \frac{\Omega_w^0(1+z)^3}{\sqrt{1+b^2}}\,\sqrt{1+b^2(1+z)^2}, \tag{4.3}$$

and Ω_w^0 is the WDM density in the present-day Universe. This model can be used also to describe several fluids in the thermal contact [70][33]. According to our physical setting, the running cosmological constant [18] is exchanging energy only with the usual matter and the last has the approximate equation of state of radiation. Then the conservation law has the form

$$\rho_r' - \frac{3(1+w)}{1+z}\,\rho_r = -\rho_\Lambda', \tag{4.4}$$

where we left w to be the equation of state parameter for the sake of generality. When starting to deal with the numerical estimates, we shall set $w = 1/3$. Finally, the Hubble parameter is given by the Friedman equation

$$H^2(z) \,=\, \frac{8\pi G}{3}\big[\rho_\Lambda(z) + \rho_r(z) + \rho_w(z)\big]. \tag{4.5}$$

The solution of the system (2.34), (4.4) and (4.5) can be performed again following the pattern of [62], since the technical complications related to the presence of DM are not critical. In order to obtain $\Omega_r(z)$ one has to consider the derivative of Eq. (4.5) and then use (2.34). After this, we arrive at the equation

$$\rho_\Lambda' \,=\, \frac{\nu}{1-\nu}\big(\rho_r' + \rho_w'\big). \tag{4.6}$$

Using (4.6) in (4.4) to eliminate ρ_Λ, after some simple algebra we obtain the differential equation for $\rho_r(z)$,

$$\rho_r' - \frac{\zeta}{1+z}\rho_r \,=\, -\nu\rho_w', \tag{4.7}$$

where

$$\zeta \,=\, 3(1+w)(1-\nu). \tag{4.8}$$

Let us stress that the interaction between radiation (remember it is all usual matter in this case) and DM, is not direct, but occurs because of the running of the cosmological constant term in Eq. (6.60), parameterized by ν, and the Friedmann equation (4.5). This implicit interaction occurs regardless of the DM satisfies separate continuity equation (4.2).

[1] Here and from now on we use the notation $\Omega_i(z) = \rho_i(z)/\rho_c^0$, where $\rho_c^0 \,=\, 3H_0^2/8\pi G$.

Using Eq. (4.3) the solution of (4.7) can be found in the form

$$\Omega_r(z) \;=\; C_0(1+z)^{\zeta} - \frac{\nu\Omega_w^0(1+z)^3}{\sqrt{1+b^2}}\Big[\sqrt{1+b^2(1+z)^2} + \frac{\zeta}{3-\zeta}\;{}_2F_1(f_a,f_b;f_c;Z)\Big], \quad (4.9)$$

with

$$C_0 = \Omega_r^0 + \frac{\nu\Omega_w^0}{\sqrt{1+b^2}}\Big[\sqrt{1+b^2} + \frac{\zeta}{3-\zeta}\;{}_2F_1(f_a,f_b;f_c;-b^2)\Big]. \quad (4.10)$$

Here ${}_2F_1(f_a,f_b;f_c;Z)$ is the hypergeometric function defined as

$${}_2F_1(f_a,f_b;f_c;Z) = \sum_{k=0}^{\infty}\frac{(f_a)_k(f_b)_k}{(f_c)_k}\,\frac{Z^k}{k!}, \quad (4.11)$$

where $(f_a)_k$ is the Pochhammer symbol. In our case

$$f_a = -\frac{1}{2}, \qquad f_b = \frac{3-\zeta}{2}, \qquad f_b = \frac{5-\zeta}{2} \qquad \text{and} \qquad Z = -b^2(1+z)^2. \quad (4.12)$$

Furthermore, $\Omega_\Lambda(z)$ is directly obtained by integrating (4.6),

$$\Omega_\Lambda(z) = B_0 + \frac{\nu}{1-\nu}\,[\Omega_r(z) + \Omega_w(z)], \quad (4.13)$$

where

$$B_0 \;=\; \Omega_\Lambda^0 - \frac{\nu}{1-\nu}\,\big(\Omega_r^0 + \Omega_w^0\big). \quad (4.14)$$

Finally, the Hubble parameter can be found from the Friedmann equation,

$$H(z) \;=\; H_0\sqrt{\Omega_\Lambda(z) + \Omega_r(z) + \Omega_w(z)}. \quad (4.15)$$

To illustrate the behavior of the model we can consider the total effective equation of state. It can be obtained using the second Friedman equation,

$$-2(1+z)HH' + 3H^2 \;=\; -8\pi G P_t \;\equiv\; -8\pi G\, w_{eff}(z)\rho_t, \quad (4.16)$$

where

$$\rho_t(z) \equiv \rho_\Lambda(z) + \rho_r(z) + \rho_w(z). \quad (4.17)$$

Thus,

$$w_{eff}(z) = \frac{2H'}{3H} - 1. \quad (4.18)$$

In Fig. 4.1, we plot $w_{eff}(z)$ for the energy balance obtained by the best fit of $\chi^2_{l_1}$ and $\chi^2_{l_1} + \chi^2_{BAO} + \chi^2_{SNIa}$ (see subsections 4.3.1 and 4.3.2 below). As expected $w_{eff}(z) \to -1$ when $z \to -1$, while $w_{eff}(z) \to \frac{1}{3}$ for $z \to \infty$ [71]. When compared with the ΛCDM model with the same Ω's and CMB+BAO+SNIa combined data are used, our model fits better for small z and approaches faster to radiation dominated epoch when z increases.

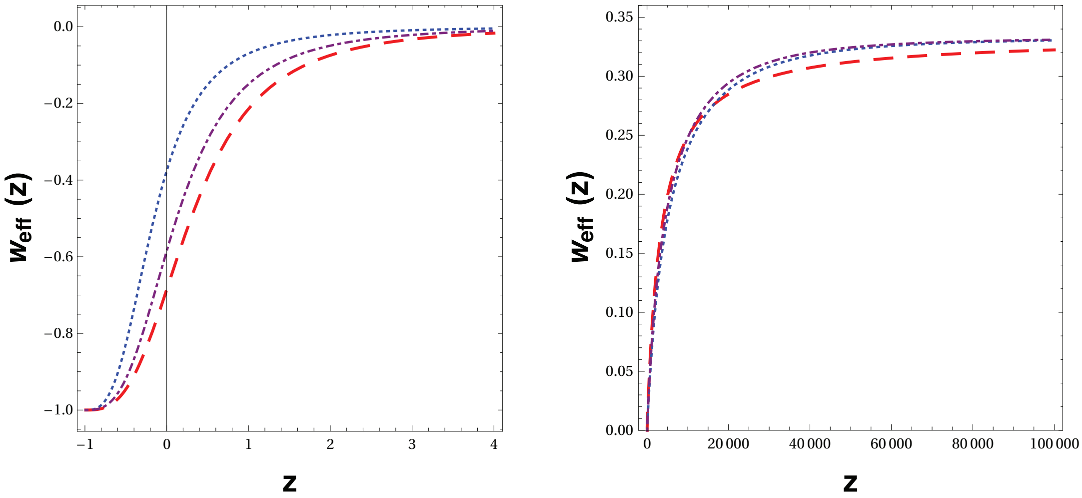

Figure 4.1: The dotted and dot-dashed lines represent our model for $\chi^2_{l_1}$ and $\chi^2_{l_1} + \chi^2_{BAO} + \chi^2_{SNIa}$ best fits, respectively, based on the results from Sec. 5.3. The dashed one represents the ΛCMD model. On the left, we plot $w_{eff}(z)$ for small values of z and in the right plot there are higher values of z. In the far future, $w_{eff}(z)$ approaches the equation of state of constant Λ. On the other hand, when $z \to \infty$, the effective equation of state approaches radiation.

4.2 Including perturbations

As it was mentioned previously in Chap.2 and Chap.3, we need to consider the perturbations

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}, \quad \rho_i \to \rho_i + \delta\rho_i, \quad U^\alpha \to U^\alpha + \delta U^\alpha, \quad V^\alpha \to V^\alpha + \delta V^\alpha \,. \tag{4.19}$$

Here U^α is the DM velocity and V^α is the usual, or baryonic, matter (radiation, in our case) velocity. In the following calculations we use the constraint $\delta U^0 = \delta V^0 = 0$ and the synchronous gauge $h_{0\mu} = 0$. The perturbation of the WDM pressure, as discussed before, is given by

$$\delta P_w = \frac{\delta\rho_w}{3}\left[1 - \left(\frac{mc^2}{\varepsilon}\right)^2\right] = \frac{\delta\rho_w(1-r)}{3}, \tag{4.20}$$

Introducing now the useful notations Eq. (4.17),

$$f_1(z) = \frac{\rho_r(z)}{\rho_t(z)}, \qquad f_2(z) = \frac{\rho_\Lambda(z)}{\rho_t(z)}, \qquad f_3(z) = \frac{\rho_w(z)}{\rho_t(z)}, \qquad g(z) = \frac{2\nu}{3H(z)}, \tag{4.21}$$

we arrive at the 00-component of Einstein equations,

$$h' - \frac{2h}{1+z} = -\frac{2\nu}{(1+z)g}\big[(1+3w)f_1\delta_r - 2f_2\delta_\Lambda + (2-r)f_3\delta_w\big], \tag{4.22}$$

Equations corresponding to the time and spatial components of the perturbation for the conservation law $\delta(\nabla_\mu T^{\mu\nu}) = 0$, have the form

$$\delta_r' + \Big[\frac{f_1'}{f_1} - \frac{3(1+w)f_2}{1+z} + \frac{(1-r-3w)f_3}{1+z}\Big]\delta_r - \frac{1+w}{(1+z)H}\Big(\frac{v}{f_1} - \frac{h}{2}\Big)$$

$$= -\frac{1}{f_1}(\delta_\Lambda f_2)' - \frac{3(1+w)f_2}{1+z}\Big[1 + \frac{(4-r)f_3}{3(1+w)f_1}\Big]\delta_\Lambda, \tag{4.23}$$

$$v' + \frac{[3(1+w)f_1 + (4-r)f_3 - 5]}{1+z}\, v = \frac{k^2(1+z)}{(1+w)H}\,(f_2\delta_\Lambda - wf_1\delta_r)\,, \tag{4.24}$$

$$\delta_w' + \Big\{\frac{f_3'}{f_3} + \frac{3(1+w)f_1 + (r-4)(f_1+f_2)}{1+z}\Big\}\delta_w + \frac{4-r}{3H(1+z)}\Big(\frac{h}{2} - \frac{u}{f_3}\Big) = 0, \tag{4.25}$$

$$u' + \Big[\frac{3(1+w)f_1 + (4-r)f_3 - 5}{1+z} - \frac{r'}{4-r}\Big]u + \frac{k^2(1+z)f_3}{H}\Big(\frac{1-r}{4-r}\Big)\delta_w = 0. \tag{4.26}$$

Here we used the notations $v = f_1\nabla_i(\delta V^i)$ and $u = f_3\nabla_i(\delta U^i)$ for divergences of the peculiar velocities and we rewrote all the previous perturbation equations in the Fourier space, using

$$f(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\, f(k,t)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \text{with} \quad k = |\mathbf{k}|. \tag{4.27}$$

Perturbing the formula (2.64) for this case, one finds

$$\delta_\Lambda = \frac{g}{f_2}\Big(\frac{v}{f_1} - \frac{h}{2}\Big). \tag{4.28}$$

The last equation is not dynamical, representing a constraint that can be replaced into other equations. Using (4.28) in (4.22), (4.23), and (4.24), we arrive at the equations

$$h' + \frac{2(\nu-1)}{1+z}\,h = \frac{2\nu}{1+z}\Big[\frac{2v}{f_1} - (1+3w)\frac{f_1}{g}\delta_r - (2-r)\frac{f_3}{g}\delta_w\Big], \tag{4.29}$$

$$\delta_r' + \Big[\frac{f_1'}{f_1} - \frac{3(1+w)f_2}{1+z} + \frac{(1-r-3w)f_3}{1+z}\Big]\delta_r = \frac{1}{f_1}\Big(\frac{gh}{2} - \frac{gv}{f_1}\Big)'$$

$$+ \frac{1+w}{1+z}\Big[3g + \frac{(4-r)gf_3}{(1+w)f_1} - \frac{1}{H}\Big]\Big(\frac{h}{2} - \frac{v}{f_1}\Big), \tag{4.30}$$

$$v' + \Big\{\frac{[3(1+w)f_1 + (4-r)f_3 - 5]}{1+z} - \frac{k^2g(1+z)}{(1+w)Hf_1}\Big\}v$$

$$= -\frac{k^2g(1+z)}{2(1+w)H}\Big(h + \frac{2wf_1}{g}\,\delta_r\Big), \tag{4.31}$$

such that the complete system of perturbation equations includes (5.1), (5.2), (5.3), (5.4) and (5.5).

4.3 Observational tests

The free parameters of the cosmological model for the early Universe with running cosmological constant and energy exchange between vacuum and matter can be constrained from various observational tests. Thus, the general framework of the model formulated above may have different applications (one can see e.g. [32] for the possibilities in a simpler model without cosmological constant running). As a first step, in the present section we consider the two tests, namely the position of the first acoustic peak of the CMB power spectrum and the inclusion of SNIa+BAO combined data.

Let us note that the process of cosmological constant decay into normal particles, as discussed in the previous sections, is effective in the primordial Universe, that is long before BBN. For this reason, we are allowed to use the transfer function in the usual standard format. However, this process leaves traces for the later epochs of the Universe evolution, encoded in the values of parameters ν and b. In this way, one can use the tests from the late phase of the Universe for exploring the effect of running cosmological constant in the earlier epoch.

The statistical analysis of the data starts with the χ^2 functions, constructed according to the general expression

$$\chi^2(X^j) = \sum_{i=1}^{N} \left[\frac{\mu_i^{obs} - \mu_i^{th}(X^j)}{\sigma_i} \right]^2 , \tag{4.32}$$

where N is the total number of observational data, μ_i^{th} are the theoretical predictions depending on free parameters X^j, and μ_i^{obs} represent the observational values with an error bar given by σ_i. In our case the free parameters are ν, Ω_w^0 and b. Let us remember that ν defines the running of the vacuum energy, while Ω_w^0 and b describe the DM relative density and warmness. As usual, $\Omega_\Lambda^0 = 1 - \Omega_w^0 - \Omega_b^0 - \Omega_r^0$. It is worth mentioning that here we are dealing with the late Universe, hence usual matter (baryonic) and radiation contents are separated.

The probability distribution function is constructed from χ^2 as

$$P(X^j) = Ae^{-\chi^2(X^j)/2}, \tag{4.33}$$

where A is a normalization constant.

4.3.1 The first CMB peak

The position of the first peak in the CMB spectrum l_1 is related to the acoustic scale l_A by the relation

$$l_1 = l_A(1 - \delta_1), \qquad \text{where} \qquad \delta_1 = 0.267 \left(\frac{\bar{r}}{0.3} \right)^{0.1} , \tag{4.34}$$

with $\bar{r} = \frac{\rho_r(z_{ls})}{\rho_m(z_{ls})}$ is evaluated at the redshift of the last scattering surface, $z_{ls} = 1090$ [72]. The acoustic scale is defined by

$$l_A = \frac{\pi \int_0^{z_{ls}} \frac{dz}{H(z)}}{\int_{z_{ls}}^{\infty} \frac{c_s(z)}{c} \frac{dz}{H(z)}}, \tag{4.35}$$

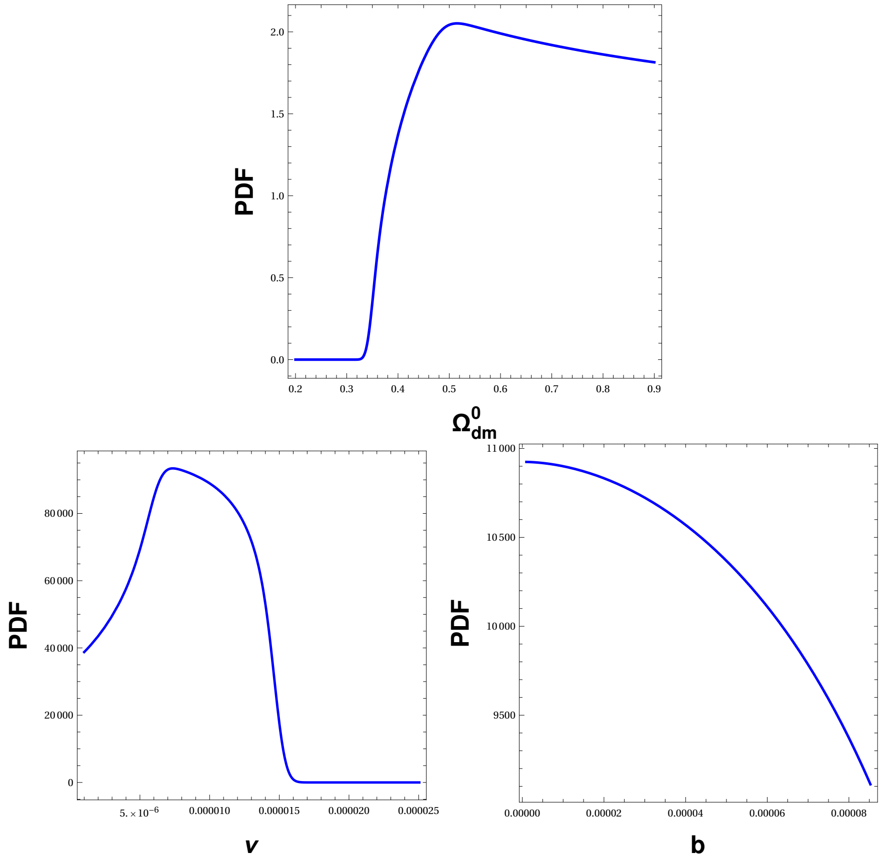

Figure 4.2: The first CMB peak one-dimensional probability distribution, after marginalizing on the other variables.

where $c_s(z)$ is the sound speed

$$c_s(z) = c\left(3 + \frac{9}{4}\frac{\Omega_b^0}{\Omega_\gamma^0 z}\right)^{-1/2}. \tag{4.36}$$

Here Ω_b^0 and Ω_γ^0 stand for the present density parameters of usual (baryonic) matter and photons, respectively. The relation (4.34) does not depend on the dark energy model. Here we consider the estimate $l_1 = 220.6 \pm 0.6$ and we use the values $\Omega_\gamma^0 = 2.47 \times 10^{-5}/h^2$, $\Omega_b^0 = 0.022/h^2$ and $\Omega_r^0 = 4.18 \times 10^{-5}/h^2$ with the reduced Hubble constant $h = 0.6732$ [46]. Furthermore, we let the free parameters run in the intervals $\nu \in \big(0,\, 10^{-4}\big)$, $b \in \big(0,\, 10^{-4}\big)$ and $\Omega_w^0 \in \big(0,\, 0.95\big)$. The minimization of the χ^2 statistics is done according to

$$\chi^2_{l_1} = \left[\frac{220.6 - l_1(\Omega_w^0, \nu, b)}{0.6}\right]^2, \tag{4.37}$$

where this function has a local minimum around

$$\Omega_w^0 = 0.550, \qquad \nu = 1.130 \times 10^{-5}, \qquad b = 4.117 \times 10^{-5}. \tag{4.38}$$

Here we can see that the current DM energy density value Ω_w^0 is higher than expected, indicating the necessity of a more robust observational test to get a better fit with respect to the standard model of cosmology (see Sec. 4.3.2).

It is easy to note that the value of Ω_w^0 quoted in (4.38) is dramatically different from the optimized value in ΛCDM. Certainly, this is not what should be expected taking the relatively small values of DM warmness and running into account. Indeed, the difference can be understood by the fact that it corresponds only to the one particular observable, namely the first CMB peak. In this special situation, $b \neq 0$ (that indicates a WDM) implies that more matter is required to reproduce the observed matter agglomeration. In what follows, we will use a more complete set of the observational data. Then the lower values for b and ν will be obtained, implying also a lower value for Ω_w^0, much closer to the conventional optimized value. In particular, taking both b and ν equal to zero, the usual ΛCDM results are obtained.

In Fig. 6.1 one can see the results for the one-dimensional marginalized probability distribution (PDF) for the free parameters of the model. It is easy to see that this test alone cannot constraint too much the parameters. Furthermore, the two-dimensional probability distribution, with both parameters being varied and one is integrated out, is shown in Fig. 6.2. The regions of higher probabilities in these plots are indicated by brighter tons. The PDF distribution shown in this subsection does not cover a compact and finite domain in the parameter space. This output of the numerical analysis is due to two reasons. First of all, it is due to the physical restriction on the parameters of the model which we imposed. For example, we assumed that both ν and b should be positive and Ω_w cannot be either negative, neither greater than a threshold value. Certainly, from the statistical point of view, this is odd, and hence we can not be surprised by the unconventional form of the region in the parameter space.

Second, it is known that for some specific models, a given parameter may have a non-negligible PDF for disjoint regions. Even if such a feature may look unusual, it can be found in the literature. One particular example is the predictions for the equation of state parameter α of the Generalised Chaplygin gas, where the constraints from the Integrated Sachs-Wolfe (ISW) effect implies either $\alpha \approx 0$ or $\alpha > 350$, with the limit $\alpha \to \infty$ giving results similar to $\alpha = 0$ [73].

4.3.2 Including BAO and SNIa data

To find better constraints for our free parameters, in this section it is constructed a more robust test using SNIa and BAO combined data. Thus, we shall use

$$\chi^2_{total} = \chi^2_{l_1} + \chi^2_{BAO} + \chi^2_{SNIa}, \tag{4.39}$$

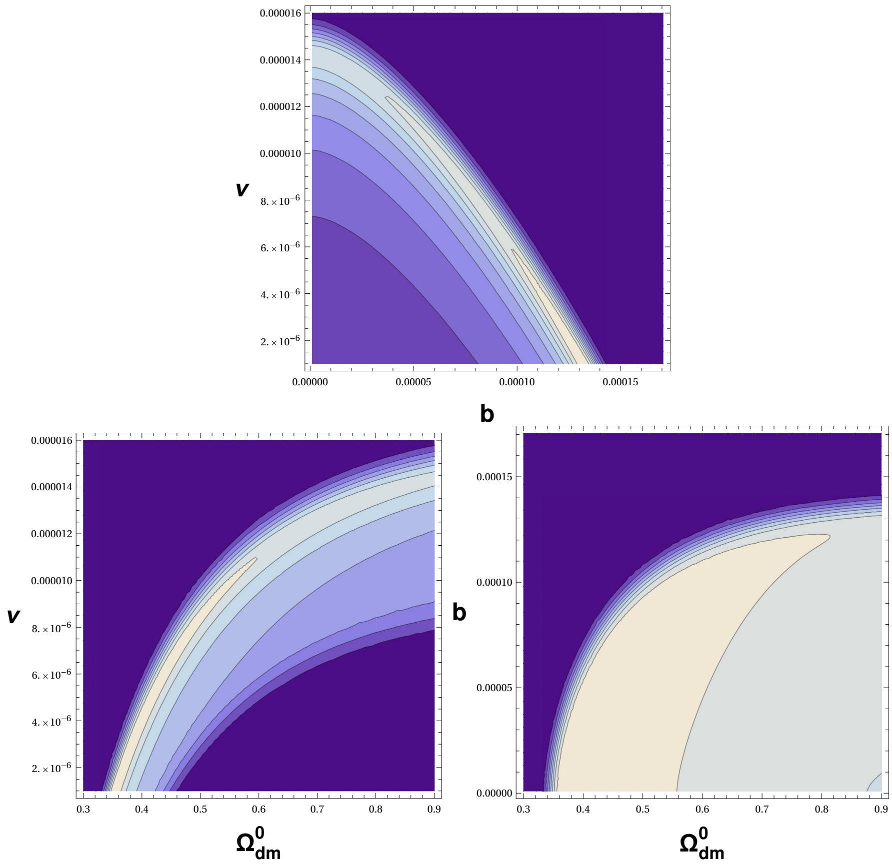

Figure 4.3: Two-dimensional probability distribution for the observational test using the first CMB acoustic peak. The brighter regions have higher probabilities.

where χ^2_{BAO} and χ^2_{SNIa} are constructed following the reference [74]. The results of this test are summarized in Table 4.2 and are given by

$$\bar{\Omega}^0_m = 0.321, \qquad \bar{\nu} = 2.442 \times 10^{-6}, \qquad \bar{b} = 1.888 \times 10^{-6}. \tag{4.40}$$

Note that this Ω^0_m is lower than the previous estimate (4.38) and therefore looks closer to the constraints obtained with the ΛCDM model. It is also important to note that in this combined test, it was used $\Omega^0_m = \Omega^0_w + \Omega^0_b$, instead of Ω^0_w as in section 2. Additionally, we observe considerable variations in ν and b values.

On the other hand, in Tables 4.1 and 4.3 we have written the best fit values for ΛCDM ($\nu = b = 0$) as our reference frame and the comparative analysis between both models. Two of the main statistical criteria to select models is the Akaike Information Criterion (AIC) [75], and the Bayesian Information Criterion (BIC) [76], defined respectively by, $AIC = \chi^2_{min} + 2\mu$ and $BIC = \chi^2_{min} + 2\mu \ln N$, where μ is the number of degree of freedom and N the number of observational data. These criteria take into account the number of free parameters of each model since the general tendency of models of higher number of free parameters is to fit better the data.

Parameter	SNIa	SNIa+BAO	SNIa+BAO+CMB
χ^2_{min}	562.227	583.289	585.440
Ω^0_m	0.261	0.275	0.308
Ω^0_Λ	0.680	0.647	0.656
AIC	566.227	587.289	589.440
BIC	587.679	608.823	610.981

Table 4.1: Summary of the observational constraints for the free parameters and for the case of $\nu = b = 0$ (ΛCDM with two free parameters).

Parameter	SNIa	SNIa+BAO	SNIa+BAO+CMB
χ^2_{min}	562.315	583.402	585.374
Ω^0_m	0.263	0.291	0.321
ν	1.970×10^{-6}	3.149×10^{-6}	2.442×10^{-6}
b	1.170×10^{-5}	1.870×10^{-5}	1.888×10^{-6}
AIC	568.315	589.402	591.374
BIC	600.493	621.703	623.685

Table 4.2: Summary of the observational constraints for the free parameters and for the model RRG+RGE (with three free parameters: Ω^0_m, ν and b).

Parameter	SNIa	SNIa+BAO	SNIa+BAO+CMB
Δ_{ik}AIC	2.088	2.113	1,934
Δ_{ik}BIC	12.814	12.880	12.704

Table 4.3: Comparative analysis between the model RRG+RGE (with three free parameters: Ω^0_m, ν and b) and the case of $\nu = b = 0$ (ΛCDM with two free parameters).

Frequently it is used on the Jeffreys scale [77] to quantify the relativity statistical relevance of the models. For the AIC (BIC) parameter, models such $\Delta AIC(\Delta BIC) < 2$ have strong support, weak support in the case $\Delta AIC(\Delta BIC) < 5$ and are disfavored for $\Delta AIC(\Delta BIC) > 10$. From table 4.3 the Jeffreys scale applied to the AIC statistical criterion favors the model RRG+RGE, while this model is strongly disfavored using the BIC criterium, a consequence of the large number of observational data, specially the SNIa data. This discrepancy in using the two criteria is a common feature found in the literature [78]-[79]. The contourplots with 1σ and 2σ levels for the CMB+BAO+SNIa combined data are shown in Fig. 4.4.

Let us note that the fact that the most probable values (4.38) and (4.40) include

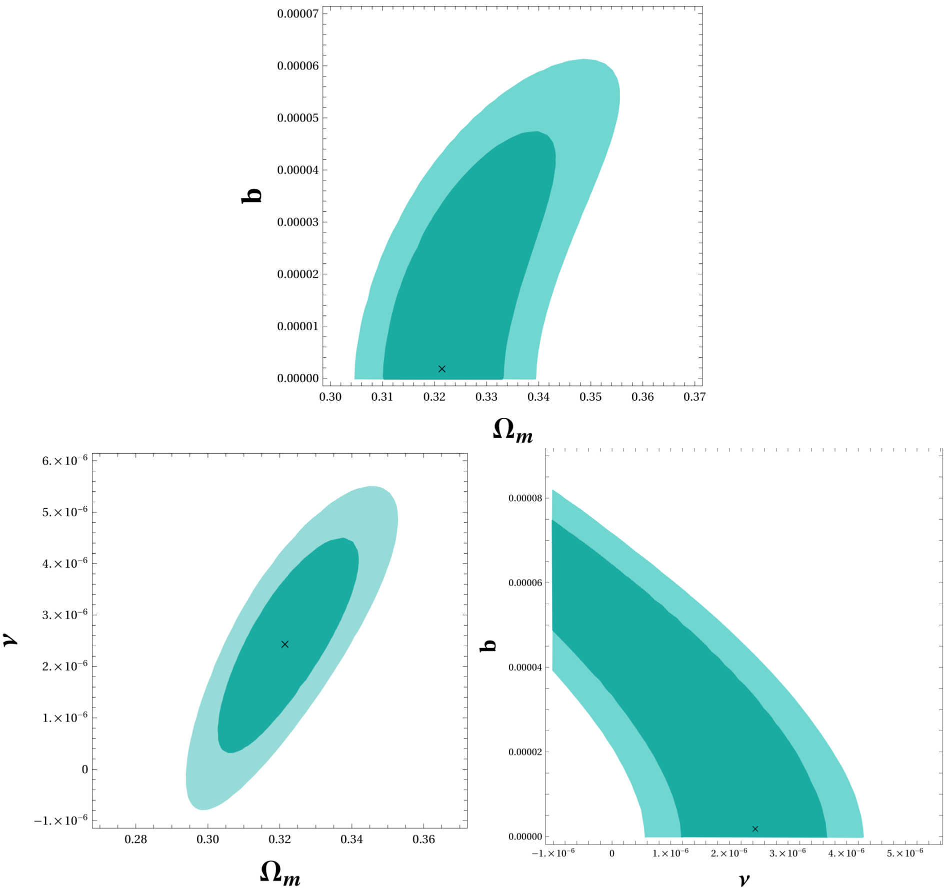

Figure 4.4: Observational constraints for our three free parameters ν, b and Ω_m^0, for 1σ and 2σ levels. Here we have used SNIa+BAO+CMB combined data. The marked points are given by $(\bar{\Omega}_m^0, \bar{b})$, $(\bar{\Omega}_m^0, \nu)$ and $(\bar{\nu}, \bar{b})$, respectively, in correspondence with best fit values presented above in Table 4.2.

$\nu \neq 0$ does not constitute proof of the running of the cosmological constant. As usual, the statistics with an extra free parameter, such as ν, always gives the best values for the non-zero parameter, and this is what we observe here. At the same time, it is remarkable that letting cosmological constant run does not lead to dramatic changes in the best fit for other parameters, such as DM relative density Ω_w^0 and the warmness b.

4.3.3 Matter power spectrum

The matter power spectrum at $z = 0$ is given by

$$P(k) = |\delta_m(k)|^2 = AkT^2(k)\Big[\frac{\bar{g}(\Omega_t^0)}{\bar{g}(\Omega_m^0)}\Big]^2, \qquad (4.41)$$

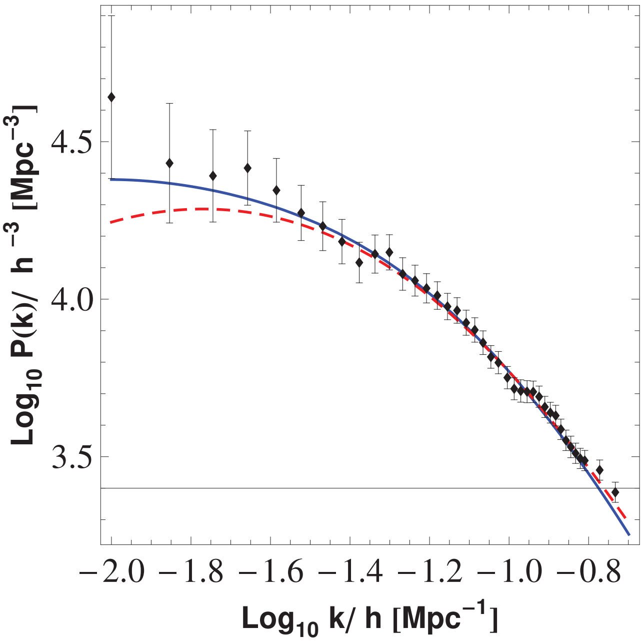

Figure 4.5: Solid line: power spectrum of our model using the best fit of $\chi^2_{l_1} + \chi^2_{BAO} + \chi^2_{SNIa}$. One can see that these values provide the linear power spectrum which is compatible with the 2dFGRS data. Dashed line: power spectrum obtained by BBKS transfer function with ΛCDM energy balance.

where A is a normalization constant of the spectrum. This constant can be fixed from the spectrum of anisotropy of the CMB radiation and

$$\bar{g}(\Omega) = \frac{5\Omega}{2\left[\Omega^{4/7} + 1.01(\Omega/2+1) - 0.75\right]}. \tag{4.42}$$

Here we use the Bardeen-Bond-Kaiser-Szalay (BBKS) transfer function [80]

$$T(k) = \frac{\ln(1+2.34q)}{2.34q}\left(1 + 3.89q + 16.1q^2 + 5.64q^3 + 6.71q^4\right)^{-1/4}, \tag{4.43}$$

where

$$q(k) = \frac{k}{h\Gamma \mathrm{Mpc}^{-1}} \qquad \text{and} \qquad \Gamma = \Omega^0_m h \exp\left\{ -\Omega^0_b - \frac{\Omega^0_b}{\Omega^0_m} \right\}, \tag{4.44}$$

to construct a set of initial conditions for the system of equations (5.1), (5.2), (5.3)-(5.5). In Fig. 4.5 we compare the data from the 2dFGRS survey [69] with the matter power spectrum of our model for the energy balance obtained by the best fit of $\chi^2_{l_1} + \chi^2_{BAO} + \chi^2_{SNIa}$ (see Table 4.2). Compared to more recent surveys (see e.g. [81]), the 2dFGRG data present the advantage of being less contaminated by the standard model used in the calibration.

Chapter 5

Including spatial curvature and some new constraints

In this chapter we shall consider the presence of spatial curvature and analyze the consequences for a cosmic epoch after recombination, this is, in a matter-dominated (MD) universe with the aim of finding some changes in the matter power spectrum and some new constraints for the free parameters ν and b with respect to Ω_k^0.

5.1 Background solution

For the sake of generality, we shall hold ω in all the expressions, but when starting the numerical estimates in Sec. 5.3, we shall set $\omega = 0$, as expected for a usual matter (in what follows we call it baryonic) component in a MD universe after recombination. The system of equations for this case, with a non-zero spatial curvature, energy exchange between cosmological constant density and baryonic matter, and adiabatically expanding ideal gas of WDM, is given by

$$H^2(z) = \frac{\kappa^2}{3}\big[\rho_\Lambda(z) + \rho_b(z) + \rho_w(z)\big] + H_0^2\Omega_k^0(1+z)^2 \tag{5.1}$$

$$\rho_b' - \frac{3(1+w)}{1+z}\,\rho_b = -\rho_\Lambda', \tag{5.2}$$

$$\rho_w' = \frac{(4-s)}{1+z}\,\rho_w. \tag{5.3}$$

In what follows we will need the total energy-momentum tensor, that is given by the sum of the baryonic, vacuum and WDM parts,

$$T^\mu_\nu = L^\mu_\nu + M^\mu_\nu, \tag{5.4}$$

where

$$\begin{aligned} L^{\mu}_{\nu} &= (1+\omega)\rho_b U^{\mu}U_{\nu} - (\omega\rho_b - \rho_{\Lambda})\delta^{\mu}_{\nu}, \\ M^{\mu}_{\nu} &= \frac{4-s}{3}\rho_w V^{\mu}V_{\nu} - \frac{1-s}{3}\rho_w \delta^{\mu}_{\nu}, \end{aligned} \tag{5.5}$$

with the associated 4-velocities U^{μ} and V^{μ}.

On the other hand, let us remember once more that one can parameterize the running of ρ_{Λ} in terms of a free parameter ν [17][16][21], as

$$\frac{d\rho_{\Lambda}}{dz} = \frac{3\nu}{8\pi G}\frac{dH^2}{dz} \tag{5.6}$$

or, equivalently, as

$$\rho_{\Lambda} = \rho^0_{\Lambda} + \frac{3\nu}{8\pi G}\left(H^2 - H_0^2\right), \tag{5.7}$$

where the sign of ν indicates whether bosons or fermions dominate in the running [21]. The set of equations formulated above, is appropriate for a simple albeit reliable description of the phase of the Universe with the running cosmological constant and the energy exchange with the matter sector. To solve the system of four equations (5.1)-(5.3) and (5.6), note first that in our model the WDM component is also decoupled, so its conservation law directly to get again

$$\Omega_w(z) = \frac{\Omega^0_w(1+z)^3}{\sqrt{1+b^2}}\sqrt{1+b^2(1+z)^2}, \tag{5.8}$$

Using (5.6) in (5.1) we get

$$\frac{d\rho_{\Lambda}}{dz} = \frac{\nu}{1-\nu}\left[\frac{d\rho_b}{dz} + \frac{d\rho_w}{dz} + \frac{6H_0^2}{\kappa^2}\,\Omega^0_k(1+z)\right] \tag{5.9}$$

and, substituting this result in (5.2), we at the simple differential equation for ρ_b,

$$\frac{d\rho_b}{dz} - \frac{\zeta}{1+z}\rho_b = -\nu\frac{d\rho_w}{dz} - \frac{6\nu}{\kappa^2}\,H_0^2\Omega^0_k(1+z), \tag{5.10}$$

where ρ_w is determined by the product $\Omega_w {\rho^0}_c$. The solution to Eq. (5.10) is given by

$$\begin{aligned} \Omega_b(z) &= C_0(1+z)^{\zeta} - \frac{2\nu}{2-\zeta}\Omega^0_k(1+z)^2 \\ &- \frac{\nu\Omega^0_w(1+z)^3}{\sqrt{1+b^2}}\Big[\sqrt{1+b^2(1+z)^2} + \frac{\zeta}{3-\zeta}\;{}_2F_1(\alpha,\beta;\gamma;Z)\Big], \end{aligned} \tag{5.11}$$

where

$$C_0 = \Omega^0_b + \frac{2\nu}{2-\zeta}\Omega^0_k + \frac{\nu\Omega^0_w}{\sqrt{1+b^2}}\Big[\sqrt{1+b^2} + \frac{\zeta}{3-\zeta}\;{}_2F_1(\alpha,\beta;\gamma;-b^2)\Big], \tag{5.12}$$

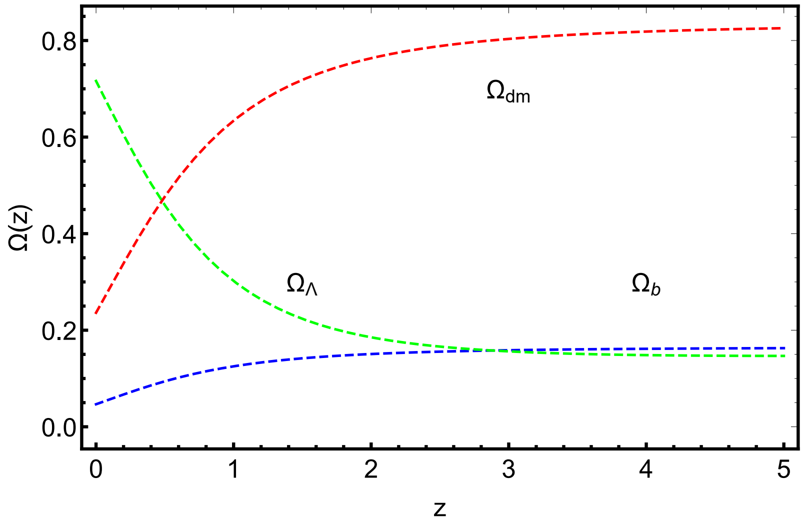

Figure 5.1: Dynamic evolution of the three energy-matter components of the Universe. One can observe the WDM domination and the domination of the running cosmological constant part for big and small redshifts z, respectively. The growing of DE as a consequence of baryons decay, it is also observed.

and ${}_2F_1(\alpha, \beta; \gamma; Z)$ is the hypergeometric function defined defined in previous chapter (see (4.11)).

The solution for ρ_Λ can be found by integrating (5.8),

$$\Omega_\Lambda(z) = B_0 + \frac{\nu}{1-\nu} \left[\Omega_r(z) + \Omega_w(z) + \Omega_k^0\, z(z+2)\right], \tag{5.13}$$

where

$$B_0 = \Omega_\Lambda^0 - \frac{\nu}{1-\nu}\left(\Omega_r^0 + \Omega_w^0\right). \tag{5.14}$$

Finally, for the square of the Hubble parameter we find

$$\begin{aligned} \left(\frac{H(z)}{H_0}\right)^2 &= 1 + \left(\Omega_b^0 + \frac{2\nu\Omega_k^0}{2-\zeta}\right)\left[\frac{(1+z)^\zeta - 1}{1-\nu}\right] + \Omega_k^0(z^2+2z)\left[1 - \frac{\nu\zeta}{(1-\nu)(2-\zeta)}\right] \\ &+ \frac{\Omega_w^0}{1-\nu}\left\{\left[\nu + \frac{\nu\zeta}{3-\zeta}\,\frac{{}_2F_1(f_a, f_b; f_c; -b^2)}{\sqrt{1+b^2}}\right](1+z)^\zeta - 1\right\} \\ &+ \frac{\Omega_w^0(1+z)^3}{\sqrt{1+b^2}}\left[\sqrt{1+b^2(1+z)^2} - \frac{\nu\zeta}{(1-\nu)(3-\zeta)}\;{}_2F_1(f_a, f_b; f_c; Z)\right]. \end{aligned} \tag{5.15}$$

In the limits of $\nu \to 0$ and $b \to 0$ we recover the standard ΛCDM model and in the case of $\omega = 0$ and $\Omega_w^0 = 0$ (that is, without WDM), we recover exactly the result presented in [1]. In Figure 6.1 is presented the cosmic evolution of the relative energy densities for normal matter, running vacuum and warm dark matter with respect to redshift z.

5.2 Cosmic perturbations

Let us consider the cosmological perturbations in this model including curvature. It proves useful once more introducing the quantities

$$f_1(z) = \frac{\rho_b(z)}{\rho_t(z)}, \qquad f_2(z) = \frac{\rho_\Lambda(z)}{\rho_t(z)}, \qquad f_3(z) = \frac{\rho_w(z)}{\rho_t(z)}, \tag{5.16}$$

and

$$g(z) = \frac{2\nu H(z)}{3H^2(z) - 3H_0^2\Omega_k^0(1+z)^2}, \tag{5.17}$$

where ρ_t is the total energy density. Thus, we arrive at the 00-component of the linearized Einstein equations,

$$h' - \frac{2h}{1+z} = -\frac{2\nu}{(1+z)g}\big[(1+3w)f_1\delta_b - 2f_2\delta_\Lambda + (2-s)f_3\delta_w\big], \tag{5.18}$$

Following the same notation for the divergences of the peculiar velocities as previous chapter, the time and spatial components of the perturbations satisfy the linear equations

$$\begin{aligned}
&\delta_b' + \Big[\frac{f_1'}{f_1} - \frac{3(1+w)f_2}{1+z} + \frac{(1-s-3w)f_3}{1+z}\Big]\delta_r - \frac{1+w}{(1+z)H}\Big(\frac{v}{f_1} - \frac{h}{2}\Big) \\
&\quad = -\frac{1}{f_1}(\delta_\Lambda f_2)' - \frac{3(1+w)f_2}{1+z}\Big[1 + \frac{(4-s)f_3}{3(1+w)f_1}\Big]\delta_\Lambda, && (5.19)\\
&v' + \frac{[3(1+w)f_1 + (4-s)f_3 - 5]}{1+z}v = \frac{k^2(1+z)}{(1+w)H}\left(f_2\delta_\Lambda - wf_1\delta_b\right), && (5.20)\\
&\delta_w' + \Big\{\frac{f_3'}{f_3} + \frac{3(1+w)f_1 + (s-4)(f_1+f_2)}{1+z}\Big\}\delta_w + \frac{4-s}{3H(1+z)}\Big(\frac{h}{2} - \frac{u}{f_3}\Big) = 0, && (5.21)\\
&u' + \Big[\frac{3(1+w)f_1 + (4-s)f_3 - 5}{1+z} - \frac{s'}{4-s}\Big]u + \frac{k^2(1+z)f_3}{H}\Big(\frac{1-s}{4-s}\Big)\delta_w = 0. && (5.22)
\end{aligned}$$

and using again the perturbation of the running equation, we arrive at the equations

$$\begin{aligned}
&h' + \frac{2(\nu-1)}{1+z}h = \frac{2\nu}{1+z}\Big[\frac{2v}{f_1} - (1+3w)\frac{f_1}{g}\delta_b - (2-s)\frac{f_3}{g}\delta_w\Big], && (5.23)\\
&\delta_b' + \Big[\frac{f_1'}{f_1} - \frac{3(1+w)f_2}{1+z} + \frac{(1-s-3w)f_3}{1+z}\Big]\delta_b = \frac{1}{f_1}\Big(\frac{gh}{2} - \frac{gv}{f_1}\Big)' \\
&\quad + \frac{1+w}{1+z}\Big[3g + \frac{(4-s)gf_3}{(1+w)f_1} - \frac{1}{H}\Big]\Big(\frac{h}{2} - \frac{v}{f_1}\Big), && (5.24)\\
&v' + \Big\{\frac{[3(1+w)f_1 + (4-s)f_3 - 5]}{1+z} - \frac{k^2 g(1+z)}{(1+w)Hf_1}\Big\}v \\
&\quad = -\frac{k^2 g(1+z)}{2(1+w)H}\Big(h + \frac{2wf_1}{g}\,\delta_b\Big). && (5.25)
\end{aligned}$$

where we shall set $w = 0$ for numerical purposes and the effect of the curvature is carried by the modified Hubble parameter H.

5.3 Some observational constraints

Let us use the equations calculated above and some of the available observational data to constrain the parameters of our model, including ν, b and Ω_k^0.

5.3.1 Supernovae Ia

In this subsection we will use the data from Supernovas Ia called "Pantheon" sample [37], which is the largest combined sample of SNIa and consists of 1048 data with the redshifts in the range $0.01 < z < 2.3$. It is a collection of the SNe Ia, discovered by the Pan-STARRS1 (PS1) Medium Deep Survey and SNe Ia from Low-z, SDSS, SNLS and HST surveys. This supernova Ia compilation uses the SALT 2 program to transform light curves into distances using a modified version of the Tripp formula [82],

$$\mu = m_B - M + \alpha x_1 - \beta c + \Delta_M + \Delta_B, \tag{5.26}$$

where μ is the distance modulus, Δ_M is a distance correction based on the host-galaxy mass of the SNIa and Δ_B is the distance correction based on predicted bias from simulations. Also, α is the coefficient of the relation between luminosity and stretch; β is the coefficient of the relation between luminosity and color and M is the absolute B-band magnitude of the fiducial SNIa with $x_1 = 0$ and $c = 0$. Also c is the color and x_1 is the light-curve shape parameter and m_B is the log of the overall flux normalization. A covariance matrix $\mathbf{C}$ is defined such that

$$\chi^2_{SNIa} = \Delta\mu^T \cdot \mathbf{C}^{-1} \cdot \Delta\mu, \tag{5.27}$$

where $\Delta\mu = \mu_{obs} - \mu_{model}$ and μ_{model} is a vector of distance modulus from a given cosmological model and μ_{obs} is a vector of observational distance modulus. The $\mu = \mathbf{m} - M$, where M is the absolute magnitude and $\mathbf{m}$ is the apparent magnitude, which is is given by

$$\mathbf{m}_{model} = M + 5Log_{10}\big(D_L\big) + 5Log_{10}\Big(\frac{c/H_0}{1Mpc}\Big) + 25 = \bar{M} + 25 + 5Log\big(D_L\big), \tag{5.28}$$

where $D_L = \frac{H_0}{c} d_L$ and $\bar{M} = M + 5Log(\frac{c/H_0}{1Mpc})$ is an nuisance parameter, which depends on the Hubble constant H_0 and the absolute magnitude M. To minimize with respect to the nuisance parameter we follow a process similar to Refs. [83][84]. Therefore, the $\chi^2_{\bar{M}marg}$ is,

$$\chi^2_{\bar{M}marg} = \tilde{a} + \log\left(\frac{\tilde{e}}{2\pi}\right) - \frac{\tilde{b}^2}{\tilde{e}}, \tag{5.29}$$

where

$$\tilde{a} = \Delta\mathbf{m}^T \cdot C^{-1} \cdot \Delta\mathbf{m}, \qquad \tilde{b} = \Delta\mathbf{m}^T \cdot C^{-1} \cdot \mathbb{I}, \qquad \tilde{e} = \mathbb{I}^T \cdot C^{-1} \cdot \mathbb{I}. \tag{5.30}$$

Here $\Delta\mathbf{m} = \mathbf{m}_{obs} - \mathbf{m}_{model}$ and $\mathbb{I}$ is the identity matrix.

Parameters	Best-fitting
Ω_w^0	0.249 ± 0.107
Ω_k^0	-0.05 ± 0.120
b	0.000655 ± 0.000300
ν	0.000407 ± 0.000070

Table 5.1: Best-fitting parameters for 1σ confidence intervals and for the χ^2_{total} including the SNIa and DR11 data sets.

5.3.2 Power Spectrum

The numerical analysis of the perturbations in our model can be confronted with the power spectrum data of the BOSS-DR11 project [5]. For the comparison of these data with the theoretical model described in the previous section, we use the chi-square statistics,

$$\chi^2_{DR11} = \sum_{i=1}^{n=37} \frac{\left[P_{the}(z_{obs,i}, k, \Omega_w^0, \Omega_k^0, b, \nu) - P_{obs,i}\right]^2}{\sigma^2_{obs,i}}, \tag{5.31}$$

where P denotes the power spectrum and we used the Planck collaboration value $\Omega_b^0 = 0.049$ [46] and $H_0 = 70\frac{km/s}{Mpc}$. In our case, the theoretical power spectrum, (P_{the}), results from the solution of the coupled system of Eqs. (5.21), (5.22) and (5.23)-(5.25). Namely, the power spectrum of the normal (baryonic) matter at the current redshift is determined as

$$P(k) = \left|\delta_b(k)\right|^2. \tag{5.32}$$

Solving system of equations (5.21), (5.22), (5.23), (5.24) and (5.25) requires specifying the initial conditions for all the variables. For this end, we follow Ref. [32] and assume the transfer function BBKS [80][85]. The first results of the numerical analysis can be seen in Table 5.1. The plot for the power spectrum which results from the approach explained above, is shown in Figure 5.2. The blue line is calculated using the best fitting given by the Table 5.1. The other lines keep all the parameters fixed while varying the parameter of the running ν (red line) or the warmness b (green line).

To determine the observational constraints using the two data sets described above, namely SNIa and matter power spectrum DR11, we define the total χ^2 as the sum of the individual contributions, in the form

$$\chi^2_{total} = \chi^2_{SNIa} + \chi^2_{DR11} \tag{5.33}$$

and elaborate this value using the data presented in this section. The results of this treatment are illustrated in Fig. 5.3, where we show how the parameters that characterize our model vary with respect to the curvature Ω_k^0. One can observe that the two data sets are complementary, except the case of the plane (Ω_k^0, b).

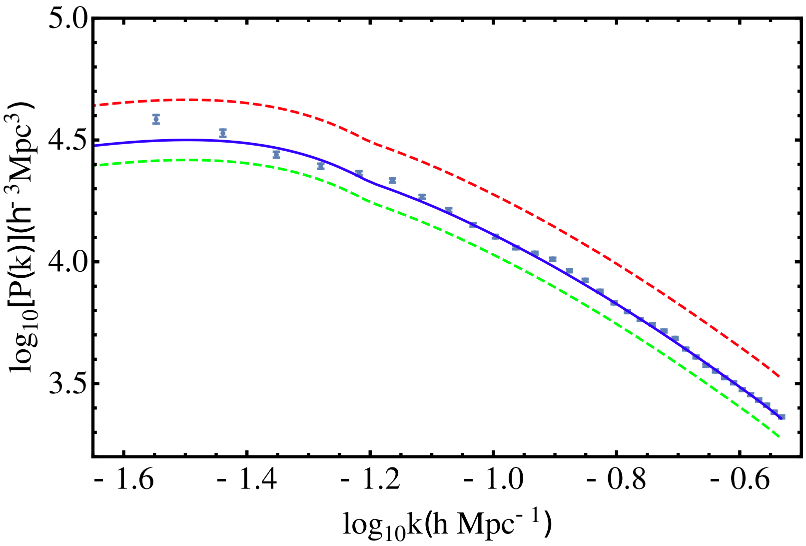

Figure 5.2: Reconstruction of the power spectrum (PS) of matter from the solution of the system of equations (5.21), (5.22), (5.23), (5.24) and (5.25). The blue line is constructed with the best fitting from Table 5.1. The green and red dashed lines correspond to the values of $b = 10^{-3}$ and $\nu = 10^{-3}$, respectively, letting fixed all the other values. It is evident that the matter PS is quite sensible to ν and b values. On the other hand, no essential changes under variation of Ω_k^0 were observed.

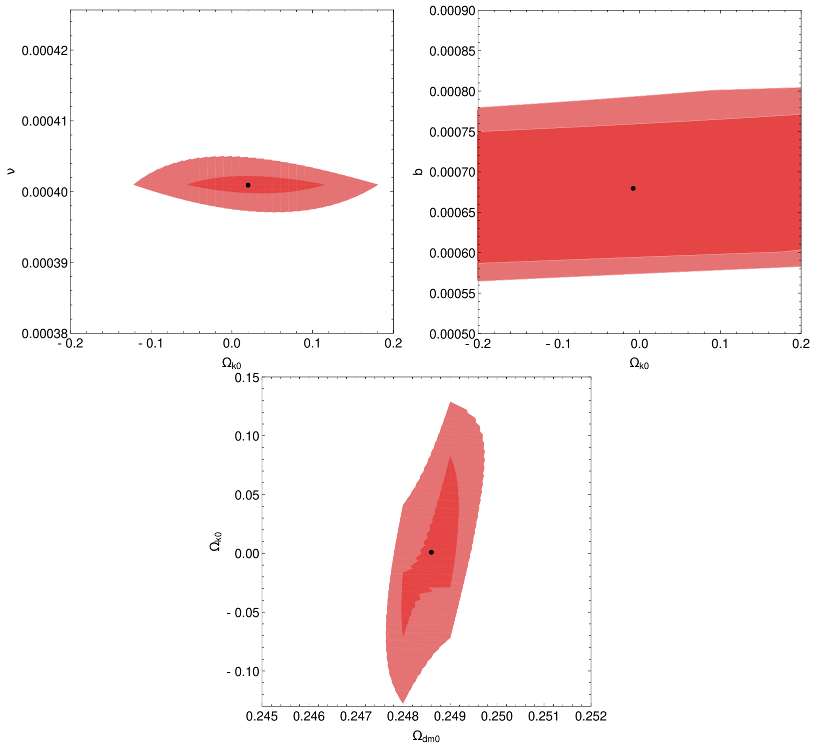

Figure 5.3: Observational constraints of the curvature parameter Ω_k^0 versus the free parameters of the model (Ω_w^0, b, ν), based on χ^2_{total}. In the top figure on the right we show the degeneracy on Ω_k^0 versus b. In the top figure on the left side there are constraints with respect to parameter ν and at the bottom we show constraints with respect to parameter Ω_w^0. In all cases we assume $H_0 = 70\frac{km/s}{Mpc}$ and $\Omega_b^0 = 0.049$.

Chapter 6

Scalar field theory for warm dark matter

In this chapter we construct a scalar field theory for the warm dark matter component modelled using again the simple reduced relativistic gas model. The main purpose is to describe the same cosmological properties of the original RRG but using the useful scalar field description, in the case of minimal coupling to gravity and conformal coupling, where the latter is constructed from the former through the properties of the conformal transformation for general scalar field actions. At the end of the chapter is also explored the possibility of including a dynamical form of dark energy introducing the running of the cosmological constant.

The RRG equation of state and other relations like continuity and warmness equations indicate that this is the model describing an ideal, this is, non-interacting gas of relativistic massive particles. Therefore, if thinking about its mapping to the theory of the scalar field, in the zero-order approximation our physical insight suggests that such a scalar model should correspond to a free scalar theory. On the other hand, since the RGG slightly deviates from the Jüttner model [31], there can be a certain defect or deviation from the free scalar field. Thus, the conformal potential should take the approximate form

$$U_{ur}(\varphi) = \frac{m^2}{2}\varphi^2 + \Delta U(\varphi) \tag{6.1}$$

where $\Delta U(\varphi)$ should be a small quantity.

6.1 Minimal scalar field

In this section, we reconstruct a general form of the scalar field potential in correspondence with the RRG. Firstly, we set up the basic relations between the scalar field and hydrodynamic variables and later we recover some basic properties of the RRG model through the equation of state in this scalar field context.

For the simple RRG cosmological model, we can use the Friedmann equations in the form

$$\rho_w = \frac{3H^2}{\kappa^2}, \qquad p_w = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot{H}\right), \tag{6.2}$$

for a spatially flat FRLW metric with line element

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{6.3}$$

where $a(t)$ is the scale factor and the dot denotes derivatives with respect to the cosmic time t. We also have the correspondent continuity equation

$$\frac{d\rho_w}{da} + \frac{(4-s)}{a}\rho_w = 0, \tag{6.4}$$

6.1.1 Reconstructing the potential

Consider the Lagrangian for a scalar field ϕ minimally coupled to gravity[1] [86]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{6.5}$$

with $\kappa^2 = 8\pi G = M_{pl}^{-2}$. The energy-momentum tensor for this scalar field is

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right]g_{\mu\nu} \tag{6.6}$$

so using again the line element (6.3), the associated energy density and pressure are given by[2]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{6.7}$$

and the conservation law $\nabla_\mu T^{\mu\nu} = 0$ yields the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{6.8}$$

We want to construct a scalar field theory which describes the same cosmological evolution as the simple model for the RRG discussed in last section, following a similar procedure as was discussed, for example, in the case of Chaplygin gas [67]. For this purpose, we need to make the equivalence

$$\rho_w = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p_w = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{6.9}$$

[1]We adopt here and what follows the next notations: ϕ is the scalar field minimally coupled to gravity and φ is the scalar field conformally coupled to gravity.

[2]When ϕ or φ appear as indices will only make reference to minimal and conformal scalar field quantities, respectively, so they are not tensor indices.

such that

$$\rho_w + p_w = \dot{\phi}^2, \qquad \rho_w - p_w = 2V(\phi). \tag{6.10}$$

Using the first of these expressions for ϕ, the Friedmann equations (6.2) and changing time derivatives by scale factor derivatives ($' \equiv d/da$), we would get

$$\left(\frac{d\phi}{da}\right)^2 = -\frac{1}{\kappa}\frac{1}{aH^2}\frac{dH^2}{da} = -\frac{1}{\kappa\, a\, \rho_w}\frac{d\rho_w}{da}, \tag{6.11}$$

and using now the conservation equation (6.4)

$$\frac{d\phi}{da} = \frac{1}{\kappa\, a}\sqrt{4-s}. \tag{6.12}$$

In order to simplify our calculations, let us to consider the redefiniton of the rest density in the form

$$\rho_d = \rho_1 a^{-3} \qquad \rightarrow \qquad \rho_d = \sqrt{3}\rho_1 a^{-3} \tag{6.13}$$

such that equation (6.12) becomes

$$\frac{d\phi}{da} = \frac{1}{2\kappa a}\left(\frac{a^2+4b^2}{a^2+b^2}\right)^{1/2}, \tag{6.14}$$

which is the fundamental and most general relation between minimal scalar field ϕ and the scale factor a, when a mapping between minimal scalar field and the RRG model is considered. Now, if we make the change of variable

$$y^2 = \frac{a^2+4b^2}{a^2+b^2}, \tag{6.15}$$

we obtain the next integral

$$\phi(y) = \frac{1}{2\kappa}\int \frac{6y^2 dy}{(y^2-1)(y^2-4)}, \tag{6.16}$$

whose solution is given by

$$\phi(y) = \frac{1}{2\kappa}\ln\left[\left(\frac{1+y}{1-y}\right)\left(\frac{2-y}{2+y}\right)^2\right], \tag{6.17}$$

or alternatively

$$\cosh 2\kappa\phi = -\frac{1}{2}\left(\frac{9y^6-15y^4+72y^2+16}{y^6-9y^4+24y^2-16}\right). \tag{6.18}$$

After some simple algebra we can find a cubic equation for y^2, this is

$$y^6 - \lambda(\phi)y^4 + \mu(\phi)y^2 + \theta(\phi) = 0, \tag{6.19}$$

whose solution, which gives y in terms of scalar field ϕ, is given by

$$y^2(\phi) = \frac{1}{3}\left[\frac{2^{1/3}\chi(\phi)}{\iota(\phi)} + \frac{\iota(\phi)}{2^{1/3}} - \lambda(\phi)\right], \tag{6.20}$$

where we have defined the functions

$$\iota(\phi) = \psi(\phi) + \sqrt{\psi^2(\phi) - 4\chi^3(\phi)}, \qquad \chi(\phi) = \lambda^2(\phi) - 3\mu(\phi), \tag{6.21}$$

$$\psi(\phi) = -2\lambda^3(\phi) + 9\lambda(\phi)\mu(\phi) - 27\theta(\phi), \tag{6.22}$$

with

$$\lambda(\phi) = \frac{3\left[1 - 12\cosh^2\kappa\phi\right]}{7 + 12\cosh^2\kappa\phi}, \qquad \mu(\phi) = \frac{24\left[1 + 4\cosh^2\kappa\phi\right]}{7 + 12\cosh^2\kappa\phi}, \tag{6.23}$$

$$\theta(\phi) = \frac{16\left[3 - 4\cosh^2\kappa\phi\right]}{7 + 12\cosh^2\kappa\phi}. \tag{6.24}$$

Therefore, we have found the solution of the scale factor a as a function of the scalar field ϕ through the auxiliary variable y, namely

$$a^2(\phi) = b^2\frac{y^2(\phi) - 4}{1 - y^2(\phi)}. \tag{6.25}$$

Additionally, solving the second equation in (6.10), we can find the potential as a function of scale factor a, this is

$$V(a) = \frac{\rho_1}{6a^4}\frac{5a^2 + 2b^2}{\sqrt{a^2 + b^2}}. \tag{6.26}$$

where substituting the solutions for $a(\phi)$ and $y(\phi)$ given by previous equation (6.25) and (6.20), we can obtain the scalar potential $V(\phi)$, and therefore, a scalar field theory for the RRG in the case of minimal coupling. In figure 6.1 (on the left) we plot this potential as a function of minimal scalar field $\kappa\phi$.

6.1.2 UR-NR limits and equation of state

In previous section we could obtain a general expression for the potential in this minimal scalar field context, but due the long and non-simple form of the function $y(\phi)$ in (6.20), the analytical treatment of this mapping can be complicated. However, we are interested in mapping our RRG model of WDM taking into account the behavior in the UR and NR limits, in order to compare our results, for instance, with the expected form of the conformal potential (see equation (6.1)). For this particular purpose, it is enough to consider the UR and NR version of fundamental relation (6.14), such that

$$UR: \quad \frac{d\phi}{da} = \frac{2}{\kappa a}, \quad \Rightarrow \quad a_{ur}(\phi) = e^{\kappa\phi/2} \quad \Rightarrow \quad V_{ur}(\phi) = \frac{\rho_2}{3}e^{-2\kappa\phi} \tag{6.27}$$

and

$$NR: \quad \frac{d\phi}{da} = \frac{1}{\kappa a}, \quad \Rightarrow \quad a_{nr}(\phi) = e^{\kappa\phi} \quad \Rightarrow \quad V_{nr}(\phi) = \frac{5\rho_1}{6}e^{-3\kappa\phi} \tag{6.28}$$

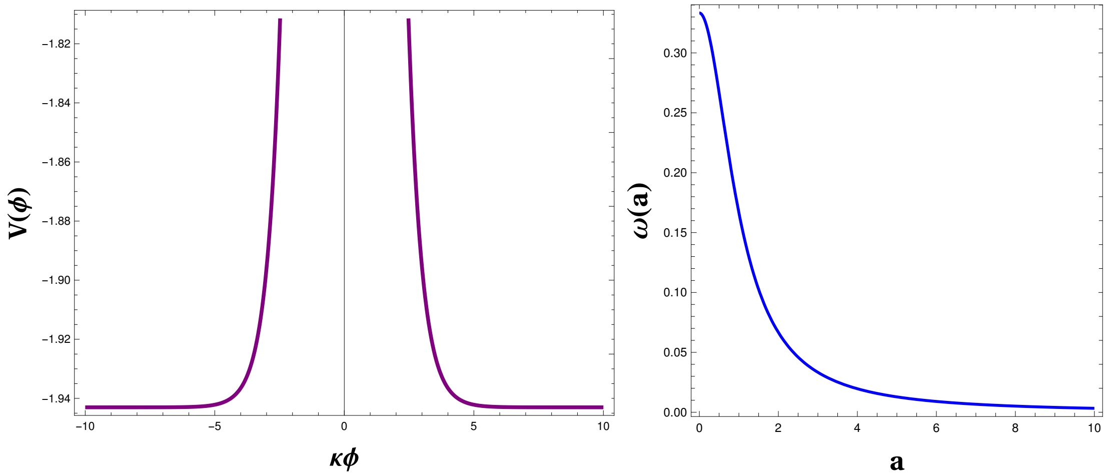

Figure 6.1: *Left panel:* The scalar potential $V(\phi)$ as a function of $\kappa\phi$. *Right panel:* The equation of state $\omega(a)$ for the minimally coupled scalar field model for the RRG where is showed the interpolation between the radiation and matter domination phases.

thus obtaining simpler expressions which can be employed in next section, when a conformal coupling for the scalar field is considered.

On the other hand, it would be interesting to check if the correspondent equation of state for this effective scalar field, it does satisfy the original properties of the equation of state for the simple RRG model, in these limits, as it was established in conditions (3.38). In this scalar context, the equation of state is given by the usual expression

$$\omega^{\phi} = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{6.29}$$

but using both Friedmann equations (6.2), we can write

$$\frac{1}{2}\dot{\phi}^2 = -\frac{1}{\kappa^2}\frac{1}{aH^2}\frac{dH^2}{da}, \qquad V(\phi) = \frac{3H^2}{\kappa^2} + \frac{a}{2\kappa^2}\frac{dH^2}{da} \tag{6.30}$$

and therefore we get

$$\omega^{\phi} = -1 - \frac{a}{3H^2}\frac{dH^2}{da} = -1 + \frac{1}{3}\left(\frac{3a^2+4b^2}{a^2+b^2}\right), \tag{6.31}$$

where it is very clear now that, for the aforementioned limits, we simply obtain

$$\omega^{\phi}_{ur} = \frac{1}{3}, \qquad \omega^{\phi}_{nr} = 0, \tag{6.32}$$

respectively. In the figure 6.1 (on the right) we have plotted the equation of state parameter in this scalar context. Additionally, using equations in (6.30) we find that that the trace of energy-momentum tensor is

$$T_\phi = 4\,\dot{\phi}^2 - V(\phi) = 3\rho_\phi\left(\omega^{\phi} - 1\right) \tag{6.33}$$

which, in the UR limit, simply vanishes, such that, we can recover the properties of the original RRG model in this minimal scalar field formalism.

6.2 Non-minimal scalar field

In this section, we explore the possibility of completing our scalar field description of the RRG model but considering a more general action, where the conformal symmetry, as a connection between minimal and non-minimal cases, it receives special attention. We use simultaneous conformal transformation and reparametrization for constructing the non-minimal (conformal) potential from the minimal case.

6.2.1 Conformal transformation and the general potential

Let us consider now the general action for a scalar field theory in the form [87]

$$\bar{S}(\phi) = \int d^4x\sqrt{-\bar{g}}\left[\bar{A}(\phi)\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \bar{B}(\phi)\bar{R} + \bar{C}(\phi)\right] \tag{6.34}$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are arbitrary functions of ϕ and consider also the simultaneous conformal transformation with arbitrary field $\sigma = \sigma(\varphi)$ and scalar field reparametrization $\phi = \phi(\varphi)$, where

$$\bar{g}^{\mu\nu} = g^{\mu\nu}e^{-2\sigma}, \qquad \sqrt{-\bar{g}} = \sqrt{-g}e^{4\sigma} \tag{6.35}$$

$$\bar{R} = e^{-2\sigma}\left[R - 6\nabla^2\sigma - 6(\nabla\sigma)^2\right] \tag{6.36}$$

with

$$\nabla^2\sigma = g^{\mu\nu}\nabla_\mu\nabla_\nu\sigma, \qquad (\nabla\sigma)^2 = g^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma. \tag{6.37}$$

After applying these transformations and reparametrization, we could rewrite the action (6.34), in the new variables as

$$S(\varphi) = \int d^4x\sqrt{-g}\left\{\left[e^{2\sigma}\bar{A}(\phi)\left(\frac{d\phi}{d\varphi}\right)^2 \right.\right.$$
$$\left.\left. +6e^{2\sigma}\left[\bar{B}(\phi)\left(\frac{d\sigma}{d\varphi}\right)^2 + \frac{d\bar{B}}{d\phi}\frac{d\sigma}{d\varphi}\frac{d\phi}{d\varphi}\right]\right] g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \bar{B}(\phi)e^{2\sigma}R + e^{4\sigma}\bar{C}(\phi)\right\} \tag{6.38}$$

such that we have the equivalence relations between both actions

$$A(\varphi) = e^{2\sigma}\left\{\bar{A}(\phi)\left(\frac{d\phi}{d\varphi}\right)^2 + 6\left[\bar{B}(\phi)\left(\frac{d\sigma}{d\varphi}\right)^2 + \frac{d\bar{B}}{d\phi}\frac{d\sigma}{d\varphi}\frac{d\phi}{d\varphi}\right]\right\} \tag{6.39}$$

$$B(\varphi) = \bar{B}(\phi)e^{2\sigma}, \qquad C(\varphi) = \bar{C}(\phi)e^{4\sigma}, \tag{6.40}$$

which can be used to find the explicit form of conformal field $\sigma(\varphi)$, the reparametrization $\phi = \phi(\varphi)$ and also the $C(\varphi)$, for two given theories related by the conformal transformation [88]. In our case, let $\bar{S}(\phi)$ be the action for a minimally coupled scalar field, such that

$$\bar{A}(\phi) = -\frac{1}{2}, \qquad \bar{B}(\phi) = \alpha, \qquad \bar{C}(\phi) = -V(\phi), \tag{6.41}$$

where we left α as an arbitrary constant, which can be fixed later on. Additionally, let $S(\varphi)$ be the action for the conformally coupled scalar field model, so we demand that its corresponding functions take the explicit form

$$A(\varphi) = -\frac{1}{2}, \qquad B(\varphi) = \frac{1}{2\kappa^2} - \frac{\xi}{2}\varphi^2, \qquad C(\varphi) = -U(\varphi) \tag{6.42}$$

where we will consider

$$U(\varphi) = U_\xi(\varphi)\Big|_{\xi=1/6} \tag{6.43}$$

in order to hold the conformal symmetry of the action $S(\varphi)$. Using the relation (6.40), we can find the conformal field σ, which is given by

$$\sigma(\varphi) = \ln\left[\frac{1}{2\alpha\kappa^2}\left(1 - \frac{\kappa^2\varphi^2}{6}\right)\right]^{1/2}, \tag{6.44}$$

and using (6.39), we can solve for ϕ and to obtain its relation with φ, this is

$$\phi(\varphi) = \sqrt{3\alpha}\ln\left(\frac{1+\frac{\kappa\varphi}{\sqrt{6}}}{1-\frac{\kappa\varphi}{\sqrt{6}}}\right) = \frac{1}{\beta}\ln\left(\frac{1+\frac{\kappa\varphi}{\sqrt{6}}}{1-\frac{\kappa\varphi}{\sqrt{6}}}\right), \tag{6.45}$$

where we have taken $\alpha = 1/(3\beta^2)$ for convenience, letting β as another arbitrary constant. Finally, using the last relation in (6.40), we can find the function $U(\varphi)$, in the form

$$U(\varphi) = \frac{9\beta^4}{4\kappa^4}V(\phi(\varphi))\left(1 - \frac{\kappa^2\varphi^2}{6}\right)^2 \tag{6.46}$$

where the minimal potential $V(\phi(\varphi))$ was already determined in previous section according to equation (6.26).

6.2.2 Explicit potential in the UR limit

To obtain explicit formula for this potential, we need to substitute equation (6.45) in the solution for $a(\phi(\varphi))$ and take the result as an input into the expression for the potential (6.26). In Figure 6.2 we plot this potential as function of conformal field φ.

From the equation (6.44), it is clear that we have the condition

$$\frac{1}{2\alpha\kappa^2}\left(1 - \frac{\kappa^2\varphi^2}{6}\right) > 0, \qquad \Longrightarrow \qquad \varphi^2 < \frac{3M_p^2}{4\pi} \approx \frac{M_p^2}{4} \tag{6.47}$$

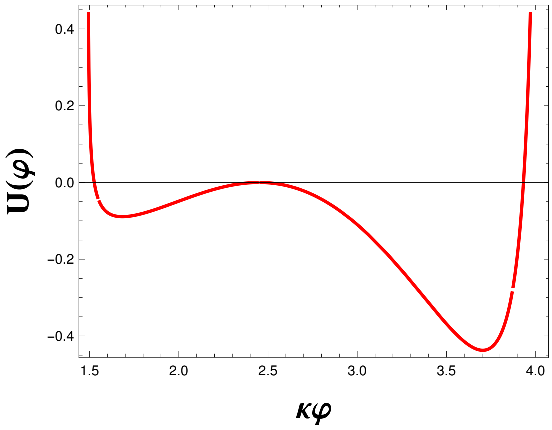

Figure 6.2: The conformal potential $U(\varphi)$ as a function of φ.

as expected for a classical scalar field. As mentioned at the end of section 2, let us to see what happen with this conformal potential once the UR limit is considered. Thus, using (3.37) in (6.14), we get again equation (6.27), namely

$$V_{ur}(\phi) = V(a_{ur}(\phi)) = \frac{\rho_2}{3}\, e^{-2\kappa\phi} \tag{6.48}$$

but using (6.45) on this UR version of the minimal potential, we get

$$V_{ur}(\varphi) \approx \frac{\rho_2}{3}\left(\frac{1-\frac{\kappa\varphi}{\sqrt{6}}}{1+\frac{\kappa\varphi}{\sqrt{6}}}\right)^{2\kappa/\beta}. \tag{6.49}$$

At this point it is possible to see that the only acceptable possibility for our arbitrary constant is $\beta = \kappa$, such that the conformal potential on this UR limit takes the form

$$U_{ur}(\varphi) = U_0(\varphi) + \Delta U(\varphi) \tag{6.50}$$

where

$$U_0(\varphi) = \frac{3\rho_2}{4}\left((\kappa\varphi)^2 - \frac{4(\kappa\varphi)}{\sqrt{6}} + 1\right), \tag{6.51}$$

and

$$\Delta U(\varphi) = \frac{3\rho_2}{4}\left(\frac{\kappa^4\varphi^4}{36} - \frac{2\kappa^3\varphi^3}{3\sqrt{6}}\right) \ll U_0(\varphi), \tag{6.52}$$

as we can easily check, due both of these terms are proportional to M_p^{-4} and M_p^{-3}, respectively, so $\Delta U(\varphi)$ is certainly small. Finally, it is possible to rewrite (6.50) in terms of an auxiliary field defined as

$$\kappa\varphi = \sqrt{\chi^2 - \frac{1}{3}} + \frac{2}{\sqrt{6}} \tag{6.53}$$

such that

$$U_{ur}(\chi) = \frac{3\rho_2}{4}\chi^2 + \Delta U(\chi) \tag{6.54}$$

also satisfying the condition

$$\Delta U(\chi) \ll \frac{\rho_2}{\kappa^2}\chi^2 \tag{6.55}$$

exactly as was expected and in perfect agreement with the above equation (6.1) in section 2.

Additionally, due the the correspondent transformation of the energy-momentum tensor and its trace, which are given by

$$T_{\mu\nu}^{(\varphi)} = e^{-2\sigma} T_{\mu\nu}^{(\phi)} \qquad \Rightarrow \qquad T^{(\varphi)} = e^{-4\sigma} T^{(\phi)} \tag{6.56}$$

in the UR limit $T^{(\varphi)}$ also vanishes, as a consequence of the above result for $T^{(\phi)}$ as given by (6.33). Therefore, we can reproduce the original UR limit for the RRG also in this non-minimal (conformal) scalar field description.

6.3 Including running vacuum energy

Let us explore the possibility of including a dark energy component in our scalar field model to unify the dark sector in a unique field. The simplest alternative is the CC, whose case only implies a constant contribution to the scalar field potential as given in (6.26) because its equation of state $p_\Lambda = -\rho_\Lambda$ does not modify the previous fundamental relation (6.14). However, the situation can be very different if a dynamical form of dark energy is considered. In previous work, the author and collaborators have considered a cosmological model for an RCC and WDM, where it was studied the evolution of the model in both background and perturbative contexts and it was compared with some cosmic observables as the first acoustic peak and the matter power spectrum [34, 64].

Let us consider the inclusion of this RCC for our scalar field model of WDM. To avoid a trivial running and to hold the WDM as a decoupled component, it is also needed to add a baryonic matter with an equation of state

$$p_b = \omega_b \rho_b \tag{6.57}$$

such that the Friedmann equations and the conservation law for this baryonic matter and RCC reads

$$\rho_w + \rho_b + \rho_\Lambda = \frac{3}{\kappa^2} H^2, \qquad p_w + \omega_b \rho_b - \rho_\Lambda = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot{H}\right) \tag{6.58}$$

$$\dot{\rho}_b + 3(1+\omega_b)\rho_b H = -\dot{\rho}_\Lambda \tag{6.59}$$

where the running of the CC is given by the renormalization group equation, this is

$$\rho_\Lambda = \rho_\Lambda^0 + \frac{3\nu}{\kappa^2}\left(H^2 - H_0^2\right) = A + BH^2 \tag{6.60}$$

with

$$A = \rho_\Lambda^0 - \frac{3\nu H_0^2}{\kappa^2}, \qquad B = \frac{3\nu}{\kappa^2}, \tag{6.61}$$

and the WDM component evolves independently as given in (6.4). In order to map this model to a minimally coupled scalar field, we can use these Friedmann equations in (6.58) to write the relations [89]

$$\dot{\phi}^2 = -\frac{2\dot{H}}{\kappa^2}, \qquad V(\phi) = \frac{3H^2}{\kappa^2} + \frac{\dot{H}}{\kappa} \tag{6.62}$$

or alternatively

$$\phi'^2 = -\frac{1}{\kappa^2 a}\frac{(H^2)'}{H^2}, \qquad V(\phi) = \frac{3H^2}{\kappa^2} + \frac{a}{2\kappa^2}(H^2)' \tag{6.63}$$

so we need to find an expression for the Hubble parameter. The general and analytical solution of this system of equations was presented in [64], where the Hubble parameter, in the case of null spatial curvature, takes the form

$$H(a)^2 = H_0^2\Big[\Omega_\Lambda^0 - \frac{\nu}{1-\nu}\left(\Omega_b^0 + \Omega_{dm}^0\right) + \frac{a^{-\zeta}}{1-\nu}\left(\Omega_b^0 + \nu\Omega_{dm}^0\right) + \frac{\Omega_{dm}^0 a^{-3}}{\sqrt{1+b^2}}\sqrt{1+\frac{b^2}{a^2}} \tag{6.64}$$

$$+\, {}_2F_1(f_a, f_b; f_c; -b^2) C a^{-\zeta} - {}_2F_1(f_a, f_b; f_c; Z) D a^{-3}\Big] \tag{6.65}$$

and we have defined the constants

$$C = \frac{\nu\zeta\Omega_{dm}^0}{(1-\nu)(3-\zeta)}, \qquad D = \frac{C}{\sqrt{1+b^2}}. \tag{6.66}$$

The complete solution for the scalar field mapping in this case, using the general expression for $H(a)$ implies two technical steps : integrating the terms evolving the hypergeometric function and inverting the resulting expression to get $a = a(\phi)$. We let the details and discussion about this full treatment for a future work. For now, and in order to compare the results with previous section 6.1, we can take the UR limit in $H(a)$, thus obtaining simpler expressions and mapping our model to a scalar field for this particular case. In this limit we can also take $|\nu| \approx 0$, such that this expression simply reduces to

$$H_{ur}(a) = H_0^2\left(\Omega_\Lambda^0 + \Omega_m^0 a^{-4}\right) \tag{6.67}$$

where $\Omega_m^0 = \Omega_b^0 + \Omega_{dm}^0$. Therefore, the first equation of the scalar field mapping yields

$$\frac{d\phi}{da} = \frac{2/\kappa a}{\sqrt{\left(\frac{\Omega_\Lambda^0}{\Omega_m^0}\right) a^4 + 1}} \tag{6.68}$$

and the potential takes the form

$$V_{ur}(a) = \frac{H_0^2}{2\kappa^2}\left(3\Omega_\Lambda^0 + \Omega_m^0 a^{-4}\right) \tag{6.69}$$

where, in order to compare with our previous result, we have to take the limit $\Omega_\Lambda^0 \to 0$ and $\Omega_b^0 \to 0$, thus obtaining exactly the expressions (6.26) and (6.27).

Conclusions

Let us conclude this thesis by summarizing our contributions and commenting them, including some perspectives and possible future work.

Summary of achievements

This thesis is devoted to the study of the cosmological model considering a running vacuum energy and the reduced relativistic gas as simple model of warm dark matter, where we report the following original results:

- It was understood that the renormalization group running of the cosmological constant, with the energy exchange between vacuum and matter can take place only in the early Universe, when the energy density of the gravitational background may be sufficient to create massive particles.
- We have implemented the model for the running of the cosmological constant in an early stage of the Universe, where the dark matter sector is modeled using the reduced relativistic gas and explore some observational consequences.
- At the background level, the model was solved analytically, taking into account the energy exchange between vacuum energy and usual (baryonic) matter. The effective equation of state parameter presented an expected evolution and we found the best fit with respect to the standard model, once the constrained values using SNIa+BAO+CMB combined data are obtained. Additionally, this effective parameter goes to radiation value faster than in the standard model for large redshift z. Besides the first CMB peak, the SNIa and BAO data are used for constraining our free parameters, obtaining a better correspondence with observations for this case.
- When considering perturbations to compute the matter power spectrum, the system of equations for the geometric perturbation, density contrasts, and velocities are found and solved numerically. We compared our results with the ones of the 2dFRG data, obtaining a better correspondence for small k, in contrast to the standard ΛCDM model.

- On the other hand, we have developed a cosmological model with a non-zero spatial curvature, running cosmological constant and warm dark matter. At the background level, we have found the analytical and general expressions for the corresponding relative energy densities, as well as the general expansion rate given by the Hubble parameter.

- The analysis of density perturbations for all involved fluids leads to the system of equations for the density contrasts. The new element compared to the previous works [1][32][34] is that this time we took into account modifications caused by the spatial curvature, including on the expansion rate. This system of equations has been solved numerically to reconstruct the corresponding matter power spectrum and to impose the restrictions on the free parameters of our model, such as the running parameter ν and the warmness b, taking into account the effect of the spatial curvature Ω_k^0.

- Once we include spatial curvature, the observational constraints, using new Pantheon Supernovae and BOSS-DR11 power spectrum data, restrict the magnitudes of both parameters ν and b at the order of 10^{-4}. The combined data prefers a relatively low warm dark matter component of the order of $\Omega_w^0 = 0.25$, as a component of the total matter balance today, $\Omega_m^0 = \Omega_w^0 + \Omega_b^0 = 0.299$. However, there is a degeneracy in the observational constraints on the parameter Ω_k^0, which can be clearly observed in the diagram showing the plane (Ω_k^0, b) in Fig. 5.3. Even though, a slight preference for a closed universe can be identified.

- Aditionally, we have studied the scalar field theory for a warm dark matter (WDM) component which is modeled using the simple reduced relativistic gas approximation (RRG). As a general result, we have found a mapping between the RRG model and a scalar field minimally coupled to gravity. Additionally, we have reconstructed the non-minimal action and potential from this minimal case.

- For the minimal coupling between gravity and the scalar field ϕ, we found the correspondent scalar potential and analyze the behavior of its analogous equation of state and the trace of the energy-momentum tensor, where the properties of the original RRG model were verified in terms of this minimal scalar field.

- In the context of non-minimal coupling to gravity, we have constructed the conformal potential from the minimal case using the properties of the conformal transformation of the metric tensor, where, as expected, we could verify the consistency of this general scalar field description in the UR limit and for the case of conformal invariance of the action $S(\varphi)$ ($\xi = 1/6$).

Comments on the running cosmological constant in a primordial universe

Our results suggest that a primordial running of the cosmological constant and the possible creation of usual (baryonic) matter particles at this early stage from vacuum energy, cannot be ruled out and deserves more detailed exploration, e.g. in the possible future work.

The model which we developed here explores the possibility that the cosmological term decays into the baryonic component in the early Universe, when the running of the cosmological constant and the intensity of the gravitational field are sufficiently strong and, on the other hand, baryonic matter contents can be regarded as ultra-relativistic particles. The parameter $\nu \neq 0$ indicates a non-constant cosmological term and the parameter b parameterizes the warmness of the matter component.

The comparison with observation points to a small deviation from the ΛCDM model as the preferred scenario, even though the strict ΛCDM case, given by $\nu = 0$, is not excluded. It must be remembered also that the running of the cosmological term implies a new free parameter in comparison to the standard cosmological model and, therefore, the results can not be interpreted such that the statistical analysis proves that the cosmological constant runs. Furthermore, the warmness of the dark matter component $b \neq 0$ is allowed, with a present-day average speed of the corresponding particles (or indefinite origin, as usual) of the order of $10^{-5}\,c$.

Let us stress now the similarities and, on the other hand, conceptual and technical differences between the model of running cosmology which we dealt with in this work and the purely phenomenological models describing the variable Dark Energy. The model developed in this paper belongs to the class of interaction models, where the energy-momentum tensor for some components does not conserve separately as it happens in the Standard Model. This means that a given component decays into another one. This class of interacting model is nowadays very popular in the study of the dark sector of the Universe, addressing some questions like the coincidence problem. However, the framework assumed here is quite different from most of these papers. In the first place, we deal with an interacting model for the early Universe, instead of a model for the late Universe. In the present case, the (dynamical) cosmological term decays into the usual (baryonic) matter when it is in the ultra-relativistic regime. On the other hand, the form of the H-dependence for the cosmological constant density in our model is defined from the quantum field theory arguments [18][17] [16]. These arguments defined the form of the IR running (6.60), leaving the unique arbitrariness in the coefficient ν.

From the technical side, it is interesting to see whether some known phenomenological models describe an energy exchange between vacuum and matter, like the one we considered here. Since there are numerous models of this sort, the complete analysis is beyond our possibilities, so we mention only one particular example.

There is some similarity with the model developed earlier in Refs. [90][91] where it was considered the energy exchange between vacuum and radiation, through the evaporation of primordial black hole. In those references the form of decaying of the cosmological term was fixed as an exponential decay, leading to a smooth transition from inflation to a radiation dominated phase, but with a prediction for the spectral index of scalar perturbations was found to contradict the observational constraints. This problem can be solved on the base of Eq. (6.60), by imposing upper bounds on the coefficient ν. On the other hand, the comparison with the scenario described above is not direct since we consider a post-inflationary phase in contrast with the case treated in the mentioned references.

Comments on the inclusion of curvature in the model of running

Adding the curvature parameter to our model of running cosmological constant with WDM, increases the number of dimensions of the parameter space, regardless the effect of space curvature is phenomenologically not very strong. Additional tests may be useful to obtain more robust constraints on the parameters ν and b. In particular, we expact that the observational constraints coming from CMB and BAO may give a strong enforcements of our results. The theoretical basis of these tests would be a natural continuation of the present work.

On the other hand, considering the generality of the model and its ability to describe different phases of the Universe, another possible perspective for further investigations work could be to study the Hubble tension by including the parameter w of baryons the equation of state, as a new free parameter and estimating the Hubble parameter today. We consider this possibility for a possible future works.

Comments on the scalar field theory for warm dark matter

Our results suggest that the RRG approximation can be successfully replaced by the scalar models, in terms of simple minimal and conformal scalar fields.

We also explore the possibility of making a similar map including a dynamical dark energy component in the form of a running cosmological constant (RCC). It was possible to verify in the UR case and for the limit of null baryon and vacuum energy densities, the correspondence with the aforementioned pure WDM case. The complete mapping for this model, considering the most general expression for the Hubble parameter $H(a)$ and taking into account a non-zero running parameter ν

will be considered in future work. The scalar field mapping of this RCC could be especially interesting because its original formulation does not admit an explicitly covariant description and therefore represents a difficult challenge for the analysis of cosmic perturbations, especially CMB and gravitational waves.

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